LINEAR ALGEBRAIC GROUPS: LECTURE 6

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1. Grassmannians over Finite Fields

As seen in the Fano plane, finite fields create geometries that are quite different from our more common $\mathbb{R}$ or $\mathbb{C}$ based geometries. These tend to be connected to 'q-deformed' versions of integers, factorials, binomial coefficients and other quantities familiar from combinatorics.

Theorem. *Over the field* $\mathbb{F}_q$,

$$|Gr(n,j)| \; = \; \binom{n}{j}_q.$$

To prove this, we require a few lemmas.

Lemma 1 (q-Pascal Lemma)**.** *For any* $q \neq 0$, $\binom{n}{j}_q$ *is the only function of* $0 \leq j \leq n$ *such that:*

(1) $\binom{n}{0}_q = \binom{n}{n}_q = 1$,

(2) $\binom{n}{j}_q = \binom{n-1}{j}_q + q^{n-j}\binom{n-1}{j-1}_q$.

Proof. (Of Lemma 1) It is clear that rules (1) and (2) uniquely determine a function, just as the usual Pascal triangle built on n-choose-j does.

Recall that

$$\binom{n}{j}_q \; = \; \frac{[n]_q!}{[j]_q! \cdot [n-j]_q!}.$$

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It is clear that $\begin{pmatrix} n \\ j \end{pmatrix}_q$ obeys requirement (1). To show it obeys requirement (2), we simply compute. We have

$$\begin{aligned}
\begin{pmatrix} n-1 \\ j \end{pmatrix}_q + q^{n-j}\begin{pmatrix} n-1 \\ j-1 \end{pmatrix}_q &= \frac{[n-1]_q!}{[j]_q! \cdot [n-j-1]_q!} \; + \; q^{n-j}\frac{[n-1]_q!}{[j-1]_q! \cdot [n-j]_q!} \\
&= \frac{[n-1]_q! \cdot [n-j]_q + q^{n-j}\,[n-1]_q! \cdot [j]_q}{[j]_q! \cdot [n-j]_q!} \\
&= \frac{[n-1]_q! \cdot \big(1+q+\cdots+q^{n-j-1} + \big(q^{n-j}\big)\,(1+q+\cdots+q^{n-1})\big)}{[j]_q! \cdot [n-j]_q!} \\
&= \frac{[n-1]_q! \cdot \big(1+q+\cdots+q^{n-j-1} + q^{n-j} + \cdots + q^{n-1})\big)}{[j]_q! \cdot [n-j]_q!} \\
&= \frac{[n]_q!}{[j]_q! \cdot [n-j]_q!}.
\end{aligned}$$

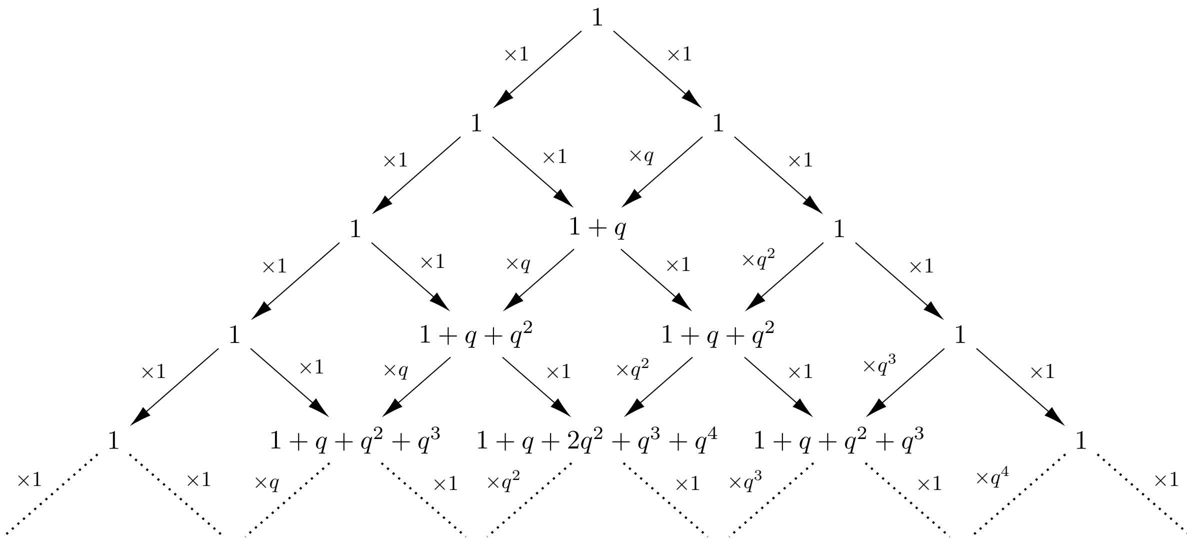
□

As mentioned in the proof, this leads to an analog of Pascal's triangle called the <u>q-Pascal triangle</u>:

$$\begin{array}{ccccccc}
 & & & \begin{pmatrix} 0 \\ 0 \end{pmatrix}_q & & & \\
 & & \swarrow & & \searrow & & \\
 & & \begin{pmatrix} 1 \\ 0 \end{pmatrix}_q & & \begin{pmatrix} 1 \\ 1 \end{pmatrix}_q & & \\
 & \swarrow & & \searrow \; \swarrow & & \searrow & \\
 & \begin{pmatrix} 2 \\ 0 \end{pmatrix}_q & & \begin{pmatrix} 2 \\ 1 \end{pmatrix}_q & & \begin{pmatrix} 2 \\ 2 \end{pmatrix}_q & \\
\swarrow & & \searrow \; \swarrow & & \searrow \; \swarrow & & \searrow \\
\begin{pmatrix} 3 \\ 0 \end{pmatrix}_q & & \begin{pmatrix} 3 \\ 1 \end{pmatrix}_q & & \begin{pmatrix} 3 \\ 2 \end{pmatrix}_q & & \begin{pmatrix} 3 \\ 3 \end{pmatrix}_q \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Numerically, we can picture the diagonals as acting almost like the usual Pascal triangle - we add two on one row to find the value centered between them on the row below.. However, the q-Pascal triangle includes multiplying by a power of q on the left diagonals, based on its position in the triangle.

Here's the triangle built to the fourth layer:



In this case, the value $\begin{pmatrix} 4 \\ 2 \end{pmatrix}_q$ is the first that doesn't look like a q-integer, or something of the form $[n]_q = 1 + q + q^2 \cdots + q^{n-1}$.

To prove our main theorem, we require one more lemma.

Lemma 2 (Grassmannian-Pascal Lemma over $\mathbb{F}_q$). *Let $\mathbb{F}_q$ be a finite field. Then*

(1) $|Gr(n, 0)| = |Gr(n, n)| = 1$,
(2) $|Gr(n, j)| = |Gr(n - 1, j)| + q^{n-j}|Gr(n - 1, j - 1)|$.

Proof. (Of Lemma 2) (1) is clear, as there is only one zero-dimensional or n-dimensional subspace in k^n for any field k. For (2), let j be given, and choose any $(n - 1)$-dimensional subspace in k^n. Let's call it k^{n-1}, and use the natural embedding coordinates

$$k^{n-1} \ = \ \{(x_1, x_2, \ldots, x_{n-1}, 0) \in k^n\} \,.$$

Suppose $L \in Gr(n, j)$. There are two choices: either $L \subseteq k^{n-1}$, or it is not.

If $L \subseteq k^{n-1}$, then $L \in Gr(n - 1, j)$. If $L \not\subseteq k^{n-1}$, then it has exactly one basis element not in k^{n-1}, so

$$\dim(k^{n-1} \cap L) = j - 1.$$

In a projective sense, we can write

$$L \ = \ \left(L \cap k^{n-1}\right) + \langle(x_1, x_2, \ldots, x_{n-1}, 1)\rangle \,,$$

where the generating vector is unique up to adding vectors in $L \cap k^{n-1}$. By dimensionality arguments, we have that

$$\left|\frac{k^{n-1}}{L \cap k^{n-1}}\right| \ = \ \frac{q^{n-1}}{q^{j-1}} = q^{n-j}.$$

In the field $\mathbb{F}_q$, a subspace of dimension $n - j$ has q^{n-j} elements.

Together, there are $|Gr(n - 1, j)|$ possibilities that lie in k^{n-1}, and $q^{n-j}|Gr(n - 1, j - 1)|$ that do not. The sum in the lemma follows. □

Finally, the theorem follows from the two lemmas.

2. Pascal's Triangle and Bruhat Cells

The previous section focused on finite fields. However, there is a version of Grassmannian-Pascal lemma that applies to *arbitrary* fields:

> **Theorem.** *Let k be any field, and j, n any natural numbers with $1 \le j \le n$. Then there exists a bijection of sets*
> $$Gr(n,j) \;\cong\; Gr(n-1,j) \;+ k^{n-j} \times Gr(n-1,j-1).$$
> *Here $+$ means disjoint union, and $\times$ is Cartesian product.*

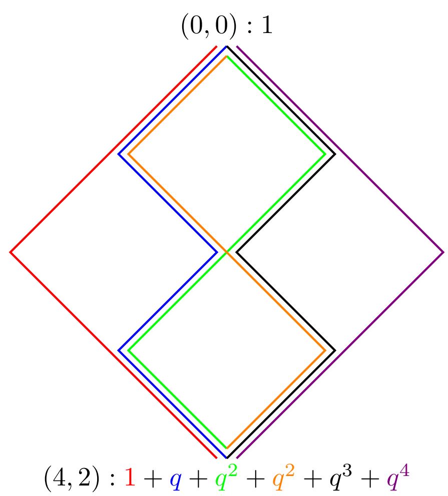
The proof is identical to the Grassmannian-Pascal lemma! This result can be used to decompose any Grassmannian into a disjoint union of copies of k^i, which are called Bruhat cells for the Grassmannian.

Example 3. Recall that
$$Gr(n,0) \;\cong\; Gr(n,n) \;\cong\; k^0,$$
and consider $Gr(4,2)$. We can decompose this as
$$\begin{aligned}
Gr(4,2) &= Gr(3,2) \;+\; k^2 \times Gr(3,1) \\
&= Gr(2,2) + k \times Gr(2,1) \;+\; k^2 \times \big(Gr(2,1) + k^2 \times Gr(2,0)\big) \\
&= k^0 + k \times (Gr(1,1) + k \times Gr(1,0)) \;+\; k^2 \times (Gr(1,1) + k \times Gr(1,0) + k^2 \times Gr(2,0)) \\
&= k^0 + k \times (k^0 + k \times k^0) \;+\; k^2 \times (k^0 + k \times k^0 + k^2 \times k^0) \\
&= k^0 + k + k^2 + k^2 + k^3 + k^4.
\end{aligned}$$

Written this way, we can see q-Pascal's triangle showing up. Each cell arises from a unique downward path from the tip of the triangle to the particular q-binomial coefficient. Reflecting a bit on this, we have

> **Theorem.** *The number of Bruhat cells in $Gr(n,j)$ is the number of distinct paths from the vertex of the q-Pascal triangle to the point (n,j). The number of such paths is the ordinary binomial coefficient* $\begin{pmatrix} n \\ j \end{pmatrix}$.

Note that steps to the right and down don't multiply (they are $\times 1$ arrows), while those to the left and down do multiply (by q^i, for a particular i). Here are all six distinct paths for $Gr(4,2)$:



If you compare this to the earlier picture of the q-Pascal triangle, you can see how each power of q arises in that triangle. If we reflect on the proof of the Grassmannian-Pascal lemma, the idea becomes clear. Suppose we take an j-dimensional subspace $L \subseteq k^n$. As we consider the intersections $L \cap k^0, L \cap k^1, L \cap k^2, \ldots,$ we see that at each stage either the dimension stays the same or increases by one. We can keep track of this using a path from the top of the q-Pascal triangle down to its (n, j) entry. Each time the dimension stays the same there is no choice involved, and we go right and down, and. Each time the dimension increases by one, there is a choice of how $L \cap k^i$ could be extended to a subspace of k^{i+1} having one extra dimension, and we go left and down. If we count all choices we make on this trip, we get the cardinality of the Bruhat cell containing L. This is q^d, where d is the dimension of that Bruhat cell.

As an exercise, prove that the dimension d can computed in the following easy way. Each Bruhat cell in $Gr(n, j)$ corresponds to a path from the top of Pascal's triangle to the (n, j) entry. To compute the dimension d of this cell, take the rectangle formed with vertices (n, j), (j, j), $(n-j, 0)$ and $(0, 0)$, and count the squares to the left of the path. Here are a few examples from the decomposition of $Gr(5, 3)$ to show the process: