Lecture 8 10-15-2018

The Walking Monad

Def - Given 2-categories C and D, a 2-functor $F: C \to D$ consists of:

- a function called F sending objects of C to objects of D
- given any 2 objects $c, c' \in C$, a functor again called F:

$$F: \hom_C(c, c') \to \hom_D(F(c), F(c'))$$

$$\begin{tikzcd} c \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & c' \end{tikzcd} \longmapsto \begin{tikzcd} F(c) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow F(\alpha)"] & F(c') \end{tikzcd}$$

preserving composition $\circ$ and identities.

Def - The <u>walking monad</u> M is the 2-category with:

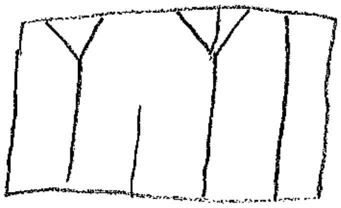
- one object $* \in M$
- $\hom_M(*, *) = \Delta_a$, the <u>augmented simplex category</u>, with:
  - objects are finite ordinals

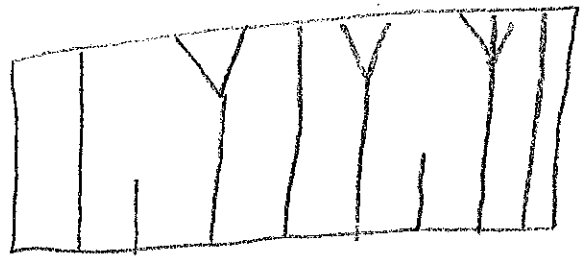
    $[0] = \{\ \}$

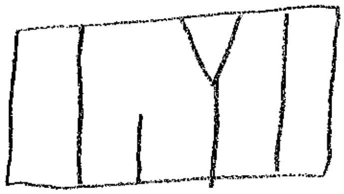
    $[1] = \{0\}$

    $[2] = \{0, 1\}$
  - morphisms are order preserving maps

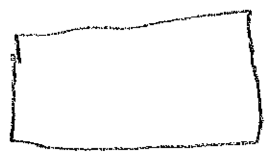
- $\circ \; \hom_M(*,*) \times \hom_M(*,*) \to \hom_M(*,*)$ looks like this

$\beta =$ 

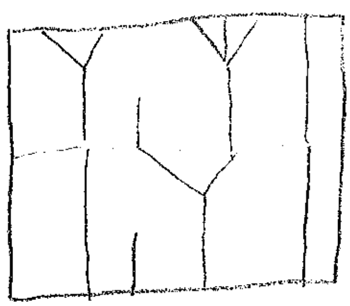
$\alpha \circ \beta =$ 

$\alpha =$ 

(If $\alpha : [m] \to [n]$, $\beta : [m'] \to [n']$, then $\alpha \circ \beta : [m+m'] \to [n+n']$)

- only possible identity: 

Note composition in the hom-category $\hom_M(*,*)$ is "vertical composition"

$\alpha\beta =$ 

$\alpha : [4] \to [4]$

$\beta : [6] \to [4]$

$\alpha\beta : [6] \to [4]$

Thm - If C is any 2-category, there's a 1-1 correspondence between monads in C and 2-functors $F : M \to C$.

Proof - Suppose we have a 2-functor $F : M \to C$; let's get a monad in C. A monad consists of $x \in C$, $T : x \to x$, $u : T \circ T \to T$, $i : 1_x \to T$ obeying assoc and left/right unit laws.

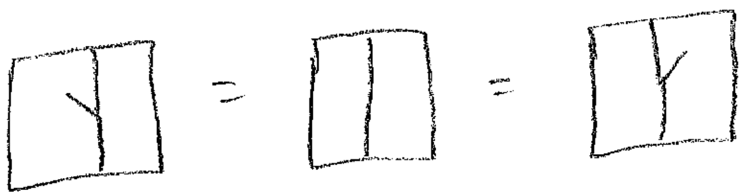
Let $x = F(*)$

$T = F([1])$

$u = F($  $)$

$i = F($  $)$

u and i obey the assoc and left/right unit laws because in M we have:

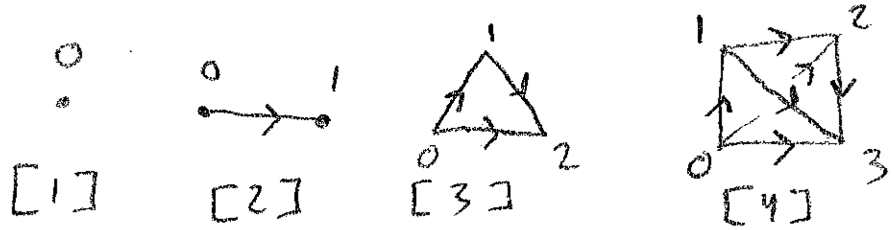


The reverse is similar. ∎

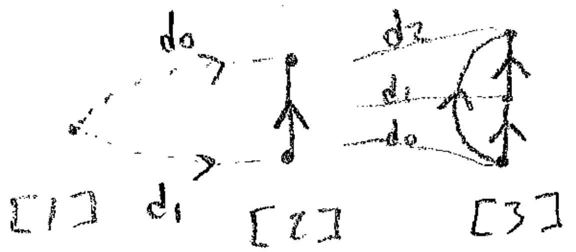
The simplex category

Remember, the simplex category Δ has nonempty finite ordinals as objects and order-preserving functions as morphisms.

Its objects can be drawn as simplexes:



Its morphisms are certain maps between simplexes:

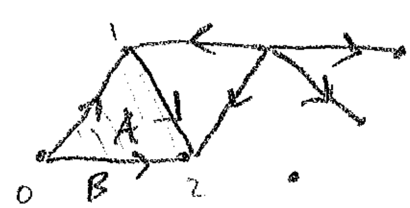


The map $d_i : [n] \to [n+1]$ is the unique order-preserving injection with i not in its range $(0 \leq i \leq n)$.

There are also other maps.

A simplicial set is a functor $F : \Delta^{op} \to Set$.
For each $n \in \mathbb{N}$, $F([n])$ is the "set of n-simplexes".
Each order-preserving map $g : [n] \to [m]$ gives a function
$F(g) : F([m]) \to F([n])$
For example, $d_i : [n] \to [n+1]$ will give a function
$F(d_i) : F([n+1]) \to F([n])$ picking out the i^{th} face of each simplex in $F([n+1])$.

So, a simplicial set could look like this:

F = 

$A \in F([3])$
$B = F(d_1)(A) \in F([2])$

Lecture 9 10-17-2018

Homology

The plan now:

$$\begin{tikzcd} \text{simplicial sets} \arrow[r] & \text{simplicial abelian groups} \arrow[r, "\sim"] & \text{chain complexes of abelian groups} \arrow[d, "H_n"', "n^{th}\text{ homology group}"] \\ & & \text{abelian groups} \end{tikzcd}$$

If a simplicial set has "holes of dimension n", they will be detected by H_n: this group won't be trivial.

has $H_1 = \mathbb{Z}$

Recall: a simplicial set is a functor $X : \Delta^{op} \to Set$.
A map of simplicial sets should thus be a natural transformation

$$\begin{tikzcd} \Delta^{op} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\alpha \Downarrow"] & Set \end{tikzcd}$$

$X([n])$ is the set of n-simplices for X, and we get maps $\alpha_{[n]} : X([n]) \to Y([n])$ which are natural.

More generally:

Def - If C is any category, a simplicial object in C is a functor $X : \Delta^{op} \to C$, and a map between these is a natural transformation

$$\begin{tikzcd} \Delta^{op} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow"] & C \end{tikzcd}$$

If C and D are categories, we write $C^D = \hom_{Cat}(D, C)$ for the category with
- functors $D \xrightarrow{F} C$ as objects
- natural transformations as morphisms

$$\begin{tikzcd} D \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=L}] & C \arrow[from=U, to=L, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

So, $C^{\Delta^{op}}$ is the category of simplicial objects in C.

Given a simplicial set $\Delta^{op} \xrightarrow{X} Set$, we can turn it into an abelian group by composing with $F : Set \to AbGrp$, the left adjoint of $U : AbGrp \to Set$. $F(X([n]))$ is the abelian group of formal $\mathbb{Z}$-linear combinations of elements of $X([n])$.

In fact, we get a functor $Set^{\Delta^{op}} \to AbGrp^{\Delta^{op}}$

on objects: $\Delta^{op} \xrightarrow{X} Set \quad \longmapsto \quad \Delta^{op} \xrightarrow{X} Set \xrightarrow{F} AbGrp$

on morphisms:

$$\begin{tikzcd} \Delta^{op} \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=L}] & Set \arrow[from=U, to=L, phantom, "\Downarrow\alpha"] \end{tikzcd} \longmapsto \begin{tikzcd} \Delta^{op} \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=L}] & Set \arrow[r, "F"] & AbGrp \arrow[from=U, to=L, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

This is called "right whiskering" and is defined to be:

$$\begin{tikzcd} \Delta^{op} \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=L}] & Set \arrow[r, bend left=50, "F"{name=U2}] \arrow[r, bend right=50, "F"'{name=L2}] & AbGrp \arrow[from=U, to=L, phantom, "\Downarrow\alpha"] \arrow[from=U2, to=L2, phantom, "\Downarrow 1_F"] \end{tikzcd}$$

Next, given a simplicial abelian group X, we'll form a chain complex:

Def - A chain complex of abelian groups, C, is a family of abelian groups and group homomorphisms

$$\begin{array}{ll} \text{2-chains} & C_2 \\ & \downarrow \partial \\ \text{1-chains} & C_1 \\ & \downarrow \partial \\ \text{0-chains} & C_0 \end{array} \qquad \text{such that } \partial \circ \partial = 0$$

A map of chain complexes $\phi : C \to D$ is:

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ C_2 \arrow[r, "f_2"] \arrow[d, "\partial_C"'] & D_2 \arrow[d, "\partial_D"] \\ C_1 \arrow[r, "f_1"] \arrow[d, "\partial_C"'] & D_1 \arrow[d, "\partial_D"] \\ C_0 \arrow[r, "f_0"] & D_0 \end{tikzcd}$$

such that the squares commute: $\partial_D f_i = f_{i-1} \partial_C$ for $i \in \mathbb{Z}^+$

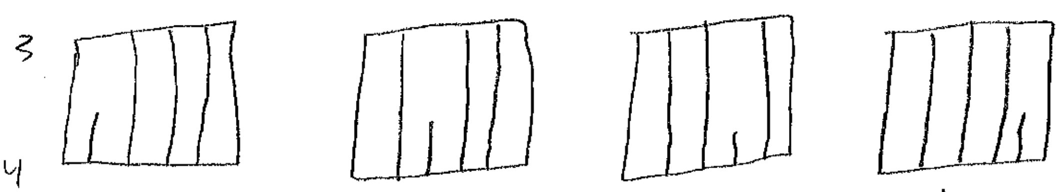
We get a category of chain complexes $Ch = Ch(AbGrp)$.

Thrm - Given $X \in AbGrp^{\Delta^{op}}$ we can form a chain complex $C(X)$ as follows.

$$C(X)_n = X([n])$$

and ∂ is defined as follows.

Recall we have morphisms $d_i : [n] \to [n+1]$ $0 \le i \le n$



d_0 $\qquad$ d_1 $\qquad$ d_2 $\qquad$ d_3.

(all the 1-1 order preserving maps $[n] \to [n+1]$)

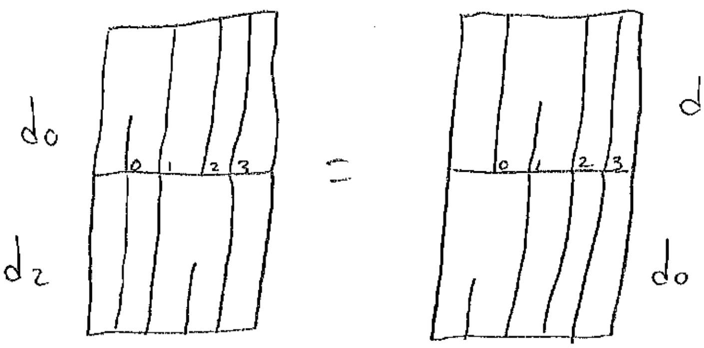
So, in Δ^{op} we have $d_i : [n+1] \to [n]$, so we have homomorphisms

$$\partial_i = X(d_i) : C(X)_{n+1} \to C(X)_n$$

and we define $\partial : C(X)_{n+1} \longrightarrow C(X)_n$ by $\partial = \sum_{i=0}^{n} (-1)^i \partial_i$

Proof - The thing to prove is $\partial \circ \partial = 0$.

$\partial \circ \partial = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} \partial_i \partial_j$ which is 0 because terms cancel in pairs using some relations that ∂_i obey.

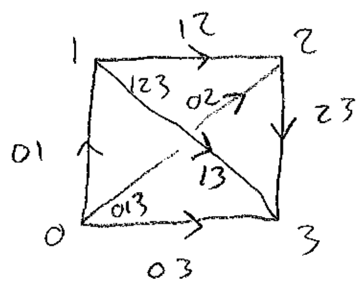


so $d_2 d_0 = d_0 d_1$ in Δ,

so $\partial_0 \partial_2 = \partial_1 \partial_0$

∎

Example: The simplicial set called "the walking 3-simplex", which looks like:



This simplicial set contains a 3-simplex 0123 and so its chain complex has a 3-chain 0123.

$$\partial(0123) = \sum_{i=0}^{3} (-1)^i \partial_i (0123) = 123 - 023 + 013 - 012$$

$$\partial^2(0123) = 23 - 13 + 12 - (23 - 03 + 02) + (13 - 03 + 01) - (12 - 02 + 01) = 0$$

Lecture 10 10-19-2018

Chain complexes from simplicial abelian groups

Thrm - There's a functor $C: \mathrm{AbGrp}^{\Delta^{op}} \to \mathrm{Ch}(\mathrm{AbGrp})$ given as follows:

- on objects: given $X: \Delta^{op} \to \mathrm{AbGrp}$, define

$C(X) \in \mathrm{Ch}(\mathrm{AbGrp})$ by:

$C(X)_n = X([n])$ (the group of n-simplices)

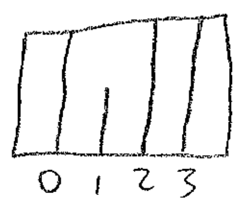
"n-chains"

Where $\partial: C(X)_{n+1} \to C(X)_n$ is given by

$\partial = \sum_{i=0}^{n} (-1)^i \partial_i$ where $\partial_i = X(d_i)$

where $d_i: [n+1] \to [n]$ in Δ^{op} corresponds to $d_i: [n] \to [n+1]$ in Δ

with $\partial^2 = 0$

$d_1 =$  in Δ

- on morphisms, given

$$\begin{tikzcd} \Delta^{op} \arrow[r, bend left=50, "X"] \arrow[r, bend right=50, "Y"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathrm{AbGrp} \end{tikzcd}$$

We define $C(\alpha): C(X) \to C(Y)$, ie $C(\alpha)_n : C(X)_n \to C(Y)_n$

$$\begin{array}{ccc} C(X)_n & \to & C(Y)_n \\ \parallel & & \parallel \\ X([n]) & \to & Y([n]) \end{array}$$

to be $\alpha_{[n]}: X([n]) \to Y([n])$

Moreover C is an equivalence, i.e. there's a functor

$D : Ch(AbGrp) \to AbGrp^{\Delta^{op}}$ such that

$$D \circ C \cong 1_{AbGrp^{\Delta^{op}}}$$

$$C \circ D \cong 1_{Ch(AbGrp)}$$

(where these are natural isomorphisms between functors)

Proof sketch: C is clearly a functor:

$$C(\beta \circ \alpha) = C(\beta) \circ C(\alpha)$$

because $C(\beta \circ \alpha)_n = (\beta \circ \alpha)_{[n]} = \beta_{[n]} \circ \alpha_{[n]}$

by def of $\circ$ for natural transformations

$= C(\beta)_n \circ C(\alpha)_n$

To get $D : Ch(AbGrp) \to AbGrp^{\Delta^{op}}$, we need to see how given $A \in Ch(AbGrp)$ we get a simplicial abelian group $D(A)$.

$$\begin{array}{c} \vdots \\ \downarrow \\ A_2 \\ \downarrow \partial \\ A_1 \\ \downarrow \partial \\ A_0 \end{array}$$

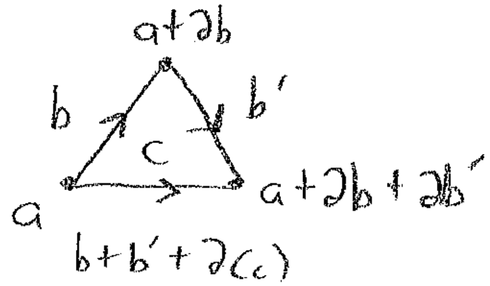
abelian group of 0-simplices $= A_0$ $\quad \bullet\, a \quad$ $a \in A_0$

abelian group of 1-simplices $= A_0 \oplus A_1$ $\quad a \xrightarrow{\ b\ } a' \quad$ $a, a' \in A_0$, $b \in A_1$

such that $\partial(b) = a' - a$

ie $a + \partial(b) = a'$

abelian group of 2-simplices $= A_0 \oplus A_1^2 \oplus A_2$



$a \in A_0$

$b, b' \in A_2$

$c \in A_3$

Check: $a + \partial(b + b' + \partial c) = a + \partial b + \partial b'$

Yes, since $\partial^2 = 0$

and so on...

Check that $C(D(A)) \cong A$ via a natural iso

Also check $D \circ C \cong 1_{AbGrp^{\Delta^{op}}}$

Given a chain complex C, note that since $\partial^2 = 0$

$\mathrm{im}(\partial : C_{n+1} \to C_n)$ "boundaries"

$\cap$

$\ker(\partial : C_n \to C_{n-1})$ "cycles"

So we can form homology groups, abelian groups

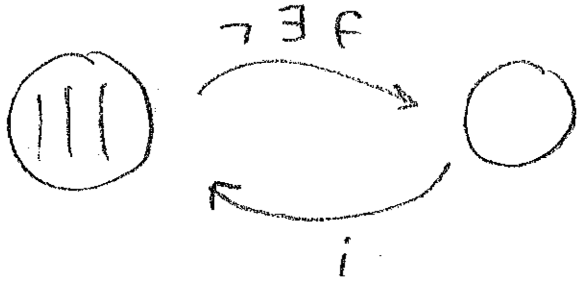
$$H_n(C) = \frac{\ker(\partial : C_n \to C_{n-1})}{\mathrm{im}(\partial : C_{n+1} \to C_n)}$$

We can write $H(C)$ for $\{H(C)_n\}_{n=0}^{\infty}$, so

$H(C)$ is an $\mathbb{N}$-graded abelian group, or

$H(C) \in \mathrm{AbGrp}^{\mathbb{N}}$.

Thrm - $H : \mathrm{Ch}(\mathrm{AbGrp}) \longrightarrow \mathrm{AbGrp}^{\mathbb{N}}$ extends to a functor.



$f \circ i = 1_{S^1}$

On H_1:

$$\begin{tikzcd} \{0\} \arrow[r, bend left, "\neg\exists H_1(f)"] & \mathbb{Z} \arrow[l, bend left, "H_1(i)"] \end{tikzcd}$$

$H_1(f) \circ H_1(i) = 1_{\mathbb{Z}}$

Lecture 10 10-22-2018

The Bar Construction

We're almost ready for something cool.

We've seen how to get the homology of a simplicial abelian group:

$$\mathrm{AbGrp}^{\Delta^{op}} \xrightarrow[C]{\sim} \mathrm{Ch}(\mathrm{AbGrp}) \xrightarrow[H]{} \mathrm{AbGrp}^{\mathbb{N}}$$

We can get simplicial abelian groups from many sources:

$\mathrm{Set}^{\Delta^{op}} \xrightarrow[F\circ -]{} \mathrm{AbGrp}^{\Delta^{op}}$ where $F: \mathrm{Set} \to \mathrm{AbGrp}$

$\mathrm{Ring}^{\Delta^{op}} \xrightarrow[U\circ -]{} \mathrm{AbGrp}^{\Delta^{op}}$ where $U: \mathrm{Ring} \to \mathrm{AbGrp}$

$\mathrm{RMod}^{\Delta^{op}} \xrightarrow[U\circ -]{} \mathrm{AbGrp}^{\Delta^{op}}$ where RMod is the category of modules over a ring R and $U: \mathrm{RMod} \to \mathrm{AbGrp}$ picks out the underlying abelian group

On the other hand, if we have adjoint functors
$$\begin{tikzcd} C \arrow[r, bend left, "F"] & D \arrow[l, bend left, "U"] \end{tikzcd}$$
we get a monad T on C

$T: C \to C$, $\mu: T\circ T \Rightarrow T$, $\eta: 1_C \Rightarrow T$ obeying assoc., l/r unit laws. Thus we get a 2-functor from the walking monad M to Cat:

$\Phi : M \to Cat$

$* \longmapsto C$

$\Delta_a = \hom_M(*,*) \longrightarrow \hom_{Cat}(C,C) = C^C$, so we get

$\Phi : \Delta_a \to C^C$ which we can restrict to the subcategory

Δ of Δ_a $\Phi : \Delta \to C^C$

Alas, $\Phi \in (C^C)^\Delta$ is a cosimplicial object in C^C

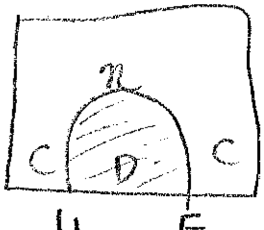
since we've got Δ instead of Δ^{op}. Let's fix this!

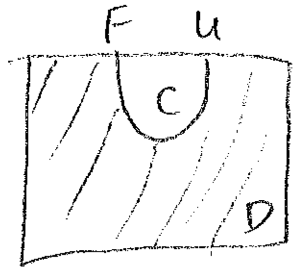
We can get a simplicial object in D^D.

Given any adjunction $C \underset{U}{\overset{F}{\rightleftarrows}} D$ in any 2-category

X, we get $1_D : D \to D$, $F \circ U : D \to D$, $F \circ U \circ F \circ U : D \to D$

and 2-morphism between these, built from





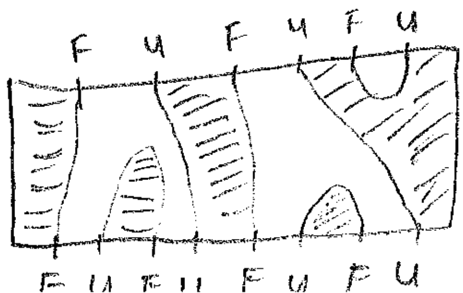
counit

unit

$\eta : 1_C \Rightarrow UF$

$\varepsilon : FU \Rightarrow 1_D$

For example:



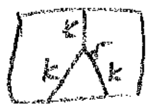
White stripes can split or end, like amoebas

(just like dark stripes can only join or start)

So, we're getting a "comod" on D, and we're getting a functor from Δ_a^{op} to $\hom_X(D, D)$.

Def - A comad on an object D in a 2-category consists of:

- a morphism $K: D \to D$
- 2-morphisms called comultiplication

$\delta: K \Rightarrow K \circ K$ 

"duplication

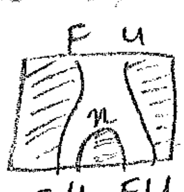
and the counit $\varepsilon: K \Rightarrow 1_D$ 

obeying <u>coassociativity</u>

and left/right <u>counit laws</u>

Thrm - An adjunction $\begin{tikzcd} C \arrow[r, bend left, "F"] & D \arrow[l, bend left, "U"] \end{tikzcd}$ in a 2-category X gives a monad on C and a comad on D.

Proof - We've done the monad; the comonad is:

$K = F \circ U$ $\quad \delta =$  $\quad \varepsilon =$ 

Check coassoc. and ℓ / r counit laws using string diagrams. $\square$

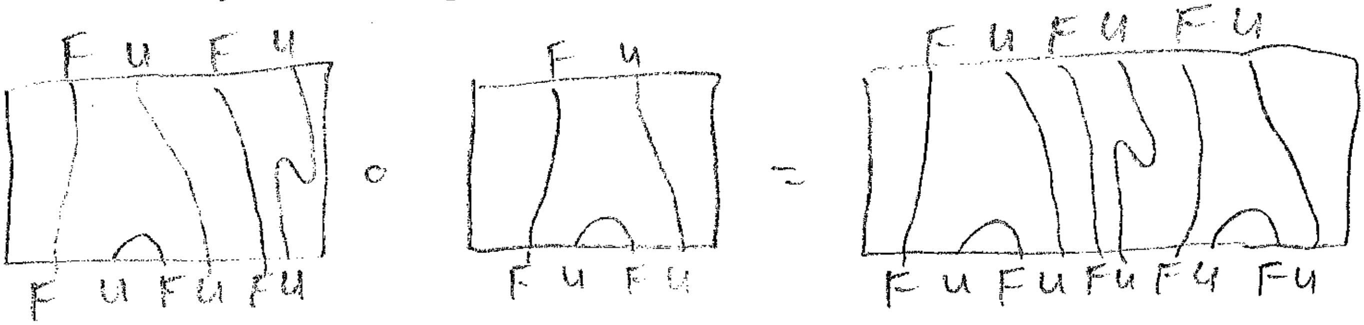
Def - The <u>walking comad</u> W is the 2-category with

- one object $*$
- $\hom_W(*,*) = \Delta_a^{op}$

with horizontal composition

$\circ : \hom_W(*,*) \times \hom_W(*,*) \longrightarrow \hom_W(*,*)$

given using string diagrams as follows:



Thm - Given any comad in a 2-cat X, there's a unique 2-functor $\Phi : W \to X$ picking out this comad.

Using these ideas we get the bar construction:

Thm - Suppose we have adjoint functors $C \underset{U}{\overset{F}{\rightleftarrows}} D$

and suppose $d \in D$. Then we get a simplicial object in D, the bar construction applied to d, $\bar{d} \in D^{\Delta^{op}}$

$$\begin{tikzcd} C \arrow[r, bend left, "F \text{ left adj}"] & D \arrow[l, bend left, "U \text{ right adj}"] \end{tikzcd} \qquad \begin{tikzcd} C^{op} \arrow[r, bend left, "F^{op} \text{ right adj}"] & D^{op} \arrow[l, bend left, "U^{op} \text{ left adj}"] \end{tikzcd}$$

If G acts on $A \in \mathrm{AbGrp}$

$A \in R\mathrm{Mod}$ where $R = \mathbb{Z}[G]$ is the group ring

Lecture 11 10-24-2018

The Bar Construction

Thrm - Given adjoint functors $\begin{tikzcd} C \arrow[r, bend left, "F"] & D \arrow[l, bend left, "U"] \end{tikzcd}$ and an object $d \in D$, there's a simplicial object in D, $\mathcal{J} \in D^{\Delta^{op}}$, given by the <u>bar construction</u>

$$\begin{tikzcd} \Delta^{op} \arrow[r, hook] \arrow[rrr, bend right, "\mathcal{J}"'] & \Delta_a^{op} \arrow[r, "\Phi"] & D^D \arrow[r, "ev_d"] & D \end{tikzcd}$$

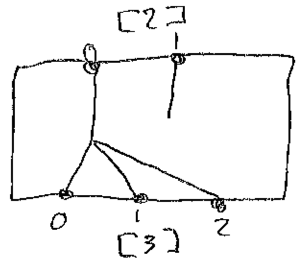
Proof - Δ, the category of non-empty finite ordinals, is contained Δ_a, the category of all finite ordinals (and order preserving maps), so we get an inclusion $\Delta^{op} \hookrightarrow \Delta_a^{op}$.

Recall that an adjunction in Cat: $\begin{tikzcd} C \arrow[r, bend left, "F"] & D \arrow[l, bend left, "U"] \end{tikzcd}$ uniquely determines a 2-functor from the walking comonad K to Cat as follows:

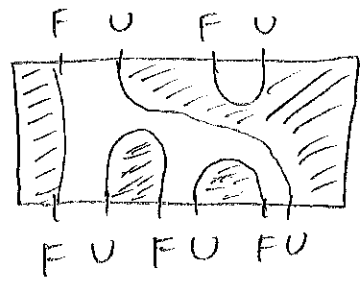
$$\Phi : K \longrightarrow Cat$$
$$* \longmapsto D$$

$$\Delta_a^{op} = hom_K(*,*) \longmapsto hom_{Cat}(D, D) = D^D$$

$f: [3] \to [2]$ is order preserving

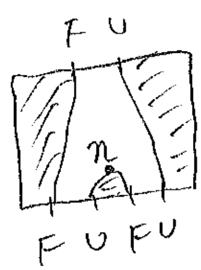


$\longmapsto$



built using

counit



comultiplication

Finally there's an evaluation functor $ev_d : D^D \to D$

$$ev_d : D^D \to D$$

$$F \mapsto F(d)$$

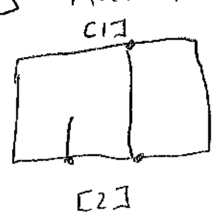
$$\alpha : F \Rightarrow G \mapsto \alpha_d : F(d) \to G(d) \quad \blacksquare$$

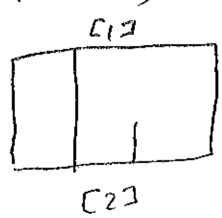
What does $\bar{d} : \Delta^{op} \to D$ really do? What does it do to objects?

$$\begin{array}{ccc} \Delta^{op} & \xrightarrow{\bar{d}} & D \\ [1] & \longrightarrow & FU(d) \\ [2] & \longrightarrow & FUFU(d) \\ [3] & \longrightarrow & FUFUFU(d) \\ \vdots & & \end{array}$$

More importantly, what does $\bar{d}$ do to morphisms?

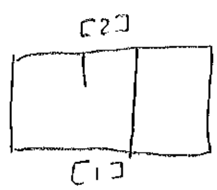
Δ has morphisms including these:

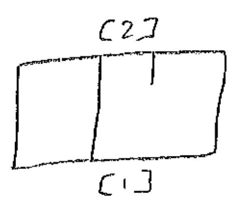
$d_0 =$ 

$d_1 =$ 

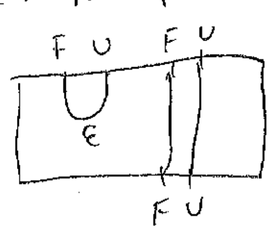
and in general, $d_0, \ldots, d_n : [n] \to [n+1]$

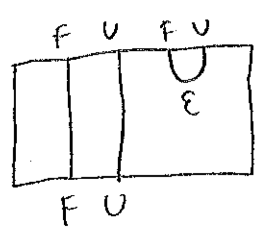
Δ^{op} thus has morphisms

$d_0 =$ 

$d_1 =$ 

Φ maps these to natural transformations:





$\varepsilon \circ 1_{FU}$ $\qquad$ $1_{FU} \circ \varepsilon$

Evaluating these on $d \in D$ we get morphisms in D:

$$FUFU(d) \xrightarrow{\varepsilon_{FU(d)}} FU(d) \qquad FUFU(d) \xrightarrow{FU(\varepsilon_d)} FU(d)$$

Example: the bar construction for G-sets

Let G be a group. A G-set is a set X with a map called an action:

$$A : G \times X \longrightarrow X$$
$$(g,x) \longmapsto A(g,x) = gx$$

obeying $g_1(g_2 x) = (g_1 g_2)x$

$$1x = x$$

There's a category of G-sets, Set^G.

There are adjoint functors

$$\begin{tikzcd} Set \arrow[r, bend left, "F"] & Set^G \arrow[l, bend left, "U"] \end{tikzcd}$$

where $F(X) = G \times X$ which is a G-set w/ $g_1(g_2, x) = (g_1 g_2, x)$

Given a G-set X, the bar construction gives a simplicial G-set $\overline{X}$ as follows:

$$\begin{tikzcd} FUFUFU(X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, phantom, "\parallel"] & FUFU(X) \arrow[r, shift left, "\varepsilon_{FU(X)}"] \arrow[r, shift right, "FU(\varepsilon_X)"'] \arrow[d, phantom, "\parallel"] & FU(X) \arrow[d, phantom, "\parallel"] \\ G\times G\times G\times X & G\times G\times X & G\times X \end{tikzcd}$$

G-set of 2-simplices	G-set of 1-simplices	G-set of 0-simplices
		

Where if S is any G-set, the counit

$$\begin{array}{l} \varepsilon_S : FU(S) \longrightarrow S \\ \qquad\quad \| \\ \qquad\quad G \times S \end{array}$$

is the action of G on S!

So, given a 1-simplex in $\bar{X}$, what are its two vertices?

$$\begin{tikzcd} \bullet \arrow[r, dash, "{(g_1,g_2,x)}"] & \bullet \end{tikzcd}$$

Answer

$$\begin{tikzcd} (g_1, g_2x) \arrow[r, dash, "{(g_1,g_2,x)}"] & (g_1g_2, x) \\ 0 & 1 \end{tikzcd}$$

In the G-set X, we have an equation

$$g_1(g_2x) = (g_1g_2)(x)$$

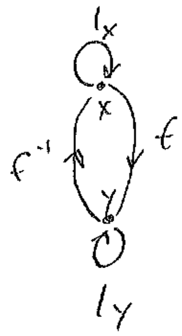
In $\bar{X}$, this equation is replaced by the above 1-simplex!

The bar construction shatters an algebraic structure and then reassembles it, replacing equations by edges!

Lecture 12 10-26-2018

The Bar Construction for G-sets

Def - A groupoid is a category where every morphism has an inverse.



Def - A monoid is a category with one object. (If $*$ is the one object, $\hom(*,*)$ has an associative operation w/ unit 1_*, so it's a monoid in the old-fashioned sense.)

Def - A group is a groupoid that's a monoid.

with $+$ as composition, this is $\mathbb{Z}_2$.

Def - Given a group G, the category of G-sets is Set^G:

- objects are functors $G \xrightarrow{A} \mathrm{Set}$, ie a set $A(*)$ with $*$ being the one object of G, and
- a function $A(g) : A(*) \to A(*)$ for each morphism in G (ie group element) such that $A(gh) = A(g) \circ A(h)$

$$A(1_*) = 1_{A(*)}$$

- morphisms are natural transformations

$$\begin{tikzcd} G \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathrm{Set} \end{tikzcd}$$

, ie a function $\alpha_* : A(*) \to B(*)$ obeying naturality.

$$\begin{tikzcd} A(*) \arrow[r, "A(g)"] \arrow[d, "\alpha_*"'] & A(*) \arrow[d, "\alpha_*"] \\ B(*) \arrow[r, "B(g)"] & B(*) \end{tikzcd}$$

commutes for all group elements g.

Thm - There are adjoint functors
$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "F"] & \mathrm{Set}^G \arrow[l, bend left, "U"] \end{tikzcd}$$

where U maps any G-set to its underlying G-set $A(*)$ and any map of G-sets
$$\begin{tikzcd} G \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Set} \end{tikzcd}$$
to its underlying function $\alpha_x : A(x) \to B(x)$, and F maps any set X to the G-set whose underlying G-set is $A(*) = G \times X$ and where group elements act as follows:

$$A(g) : G \times X \to G \times X$$
$$(h, x) \longmapsto (gh, x)$$

and given $f : X \to Y$, $F(f) : F(X) \to F(Y)$ is

$$G \times X \to G \times Y$$
$$(g, x) \mapsto (g, f(x))$$

Proof sketch: To show F is the left adjoint of U, we need a natural isomorphism
$$\hom_{G\text{-set}}(F(X), A) \xrightarrow[\sim]{\alpha} \hom_{\mathrm{Set}}(X, U(A))$$
$X \in \mathrm{Set}$
$A \in \mathrm{Set}^G$

This is: $\hom_{G\text{-set}}(\underbrace{G \times X}_{\text{shorthand for } F(X)}, \underbrace{Y}_{\text{really } A(*)}) \xrightarrow[\sim]{\alpha} \hom_{\mathrm{Set}}(X, \underbrace{Y}_{\text{as a set}})$

So given a map of G-sets $f : G \times X \to Y$, we'll choose this map of sets

$$\alpha(f) : X \to Y$$
$$x \mapsto f(1, x)$$

Conversely, given a function $f: X \to Y$,
we have a map of G-sets

$\alpha^{-1}(f): G \times X \to Y$

$(g, x) \longmapsto g f(x)$

This really is a map of G-sets:

$\alpha^{-1}(f)(h(g,x)) = \alpha^{-1}(hg, x) = (hg) f(x)$

$h \alpha^{-1}(f)(g,x) = h(g(f(x)))$. $\boxtimes$

HW: Figure out the unit and counit of this adjunction (due Wed)

$$\begin{tikzcd} Set \arrow[r, bend left, "F"] & Set^G \arrow[l, bend left, "U"] \end{tikzcd}$$

The idea: given any adjoint functors $\begin{tikzcd} C \arrow[r, bend left, "F"] & D \arrow[l, bend left, "U"] \end{tikzcd}$ we get
the unit and counit as follows:

$$\hom_D(F(c), F(c)) \xrightarrow{\alpha} \hom_C(c, UF(c))$$

unit: $\quad 1_{F(c)} \longmapsto \eta_c \qquad c \in C, \; d \in D$

$$\hom_D(FU(d), d) \xleftarrow{\alpha^{-1}} \hom_C(U(d), U(d))$$

$$\varepsilon_d \longleftarrow 1_{U(d)}$$

*counit given away last time when doing bar construction at high speed

Everything so far also works for monoids: we get $U: Set^M \to Set$
from any monoid M:

$U(A) = A(*)$ where $* \in M$ is the object.

Given a group G, classifying space of G has G as its fundamental group, and higher homotopy groups are trivial

Lecture 13 10-29-2018

The Bar Construction for Groups

Given a group G we have an adjunction
$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "F"] & \mathrm{Set}^G \arrow[l, bend left, "U"] \end{tikzcd}$$

and given any G-set X, the counit

$\varepsilon_X : F(U(X)) \to X$ is $\varepsilon_X : G \times X \to X$
$$(g, x) \longmapsto gx$$

The bar construction gives a simplicial G-set $\overline{X}$:

$$FUFUFU(X) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} FUFU(X) \substack{\xrightarrow{\varepsilon_{FU(X)}} \\ \xrightarrow{FU(\varepsilon_X)}} FU(X)$$

2-simplices $\overline{X}([3])$ 1-simplices $\overline{X}([2])$ 0-simplices $\overline{X}([1])$

where the maps are built using the counit. More concretely:

$\overline{X}([n]) = (FU)^n(X) = G^n \times X$ and we get

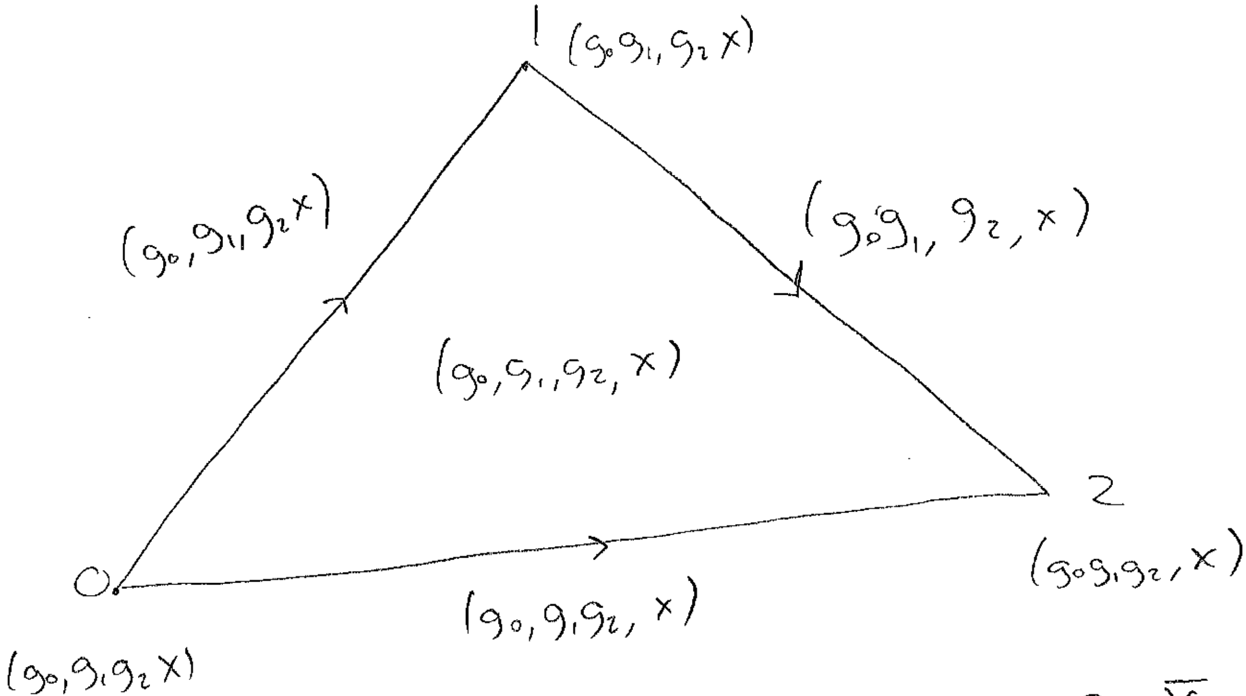
$$\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} G \times G \times G \times X \substack{\xrightarrow{\overline{X}(d_0)} \\ \xrightarrow{\overline{X}(d_1)} \\ \xrightarrow{\overline{X}(d_2)}} G \times G \times X \substack{\xrightarrow{\overline{X}(d_0)} \\ \xrightarrow{\overline{X}(d_1)}} G \times X$$

where $\overline{X}(d_i) : X([n+1]) \to X([n])$ is the map from $G^{n+1} \times X$

to $G^n \times X$ given by

$$\overline{X}(d_i)(g_0, \dots, g_n)(x) = \begin{cases} (g_0, g_1, \dots, g_i g_{i+1}, \dots, x) & 0 \le i < n \\ (g_0, \dots, g_{n-1}, g_n x) & i = n \end{cases}$$

For example:



Each element of X gives many vertices of $\overline{X}$, connected by edges. All vertices above correspond to same element of X:

$$g_0(g_1g_2x) = g_0g_1(g_2x) = (g_0g_1g_2)(x)$$

Equations in X have become edges in $\overline{X}$, "equations between equations" become triangles, etc.

Our simplicial G-set $\overline{X}$ has an underlying simplicial set:

$\overline{X}: \Delta^{op} \to Set^G$ composed w/ $U: Set^G \to Set$ gives

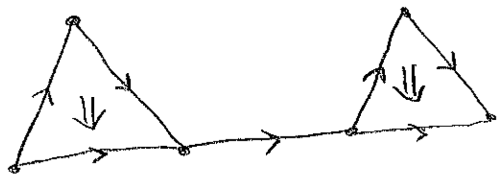
$U \circ \overline{X}: \Delta^{op} \to Set$.

Then: we can turn any simplicial set into a topological space.

Thrm - There's a functor called geometric realization

$$|\cdot| : Set^{\Delta^{op}} \to Top$$

Proof sketch: It's visually evident:



is a topological space. To prove it:

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[d] & Set^{\Delta^{op}} \arrow[dl, dotted, "|\cdot|"] \\ Top & \end{tikzcd}$$

We can turn any simplex into a space; this functor extends to geometric realization using the Yoneda embedding

$$\Delta \hookrightarrow Set^{\Delta^{op}}$$
$$[n] \longmapsto \hom(-,[n])$$

sending any simplex to the simplicial set that looks like that simplex. ▨

Thrm - If X is a G-set, the space $|U \circ \overline{X}|$ has one connected component for each element of X. Each connected component is contractible.

So we've "puffed up" X, replacing each element by a contractible connected component; this is called "cofibrant replacement".

Proof of Part 1: It's sufficient to show that if two vertices of $|U \circ \overline{X}|$, ie elements of $G \times X$, map to the same element of X by:

$$\varepsilon_X : G \times X \longrightarrow X$$

then they're connected by an edge.

Suppose (g,x) and (h,y) have $gx = hy$.

Want an edge in $\overline{X}$ connecting them:

$$\begin{tikzcd} (g,x) \arrow[r, "(g_0, g_1, z)"] & (h,y) \end{tikzcd}$$

So we want $(g_0, g_1, z) \in G \times G \times X$ such that

$$(g_0 g_1, z) = (h, y)$$

$$(g_0, g_1 z) = (g, x)$$

So need:

$g_0 g_1 = h$

$z = y$ ✓

$g_0 = g$ ✓

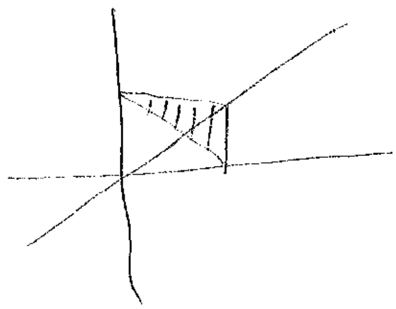
$g_1 z = x$

So $g_0 = g$, $z = y$, but what's g_1? $g_1 = g_0^{-1} h$ by first eqn, and last eqn says $g_0^{-1} h y = x \iff hy = g_0 x = gx$, which is true.

Conversely, any two guys connected by edge map to same element of X:

$(g_0, g_1 x) \qquad (g_0, g_1, x) \qquad (g_0 g_1, x)$

and both vertices map to $g_0 g_1 x \in X$.



$G^n \times X$

Lecture 14 10-31-2018

Group Cohomology

Recall: if X is a G-set, $\bar{X}$ is a simplicial G-set:

$$G\times G\times G\times X \;\substack{\xrightarrow{\bar{X}(d_0)}\\ \xrightarrow{\bar{X}(d_1)}\\ \xrightarrow{\bar{X}(d_2)}}\; G\times G\times X \;\substack{\xrightarrow{\bar{X}(d_0)}\\ \xrightarrow{\bar{X}(d_1)}}\; G\times X$$

where $\bar{X}(d_i): G^{n+1}\times X \to G^n \times X$ is

$$\bar{X}(d_i)(g_0,\dots,g_n,x) = \begin{cases} (g_0,\dots,g_ig_{i+1},\dots,x) & 0\le i<n \\ (g_0,\dots,g_{n-1},g_nx) & i=n \end{cases}$$

Note all the sets $G^n\times X$ are G-sets:

$$g(g_0,g_1,\dots,g_{n-1},x) = (gg_0,g_1,\dots,g_{n-1},x)$$

and maps $\bar{X}(d_i)$ are maps of G-sets.

If $X=\{*\}$, ie $X=1$, we call $\bar{X}$ "$\mathbb{E}G$"

$\mathbb{E}G$ looks like:

$$G\times G\times G \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G\times G \;\substack{\longrightarrow\\ \longrightarrow}\; G$$

w/ $$\bar{X}(d_i)(g_0,\dots,g_n) = \begin{cases} (g_0,\dots,g_ig_{i+1},\dots,g_n) & 0\le i<n \\ (g_0,\dots,g_{n-1}) & i=n \end{cases}$$

Def - The geometric realization $|\mathbb{E}G|$ is called $EG\in Top$, the universal contractible G-space.

Note: $|\cdot| : Set^{\Delta^{op}} \to Top$

but the bar construction gives a simplicial G-set $\overline{X}$ from any G-set, so its geometric realization is a <u>G-space</u>: a topological space on which G acts as continuous maps. In fact, we have:

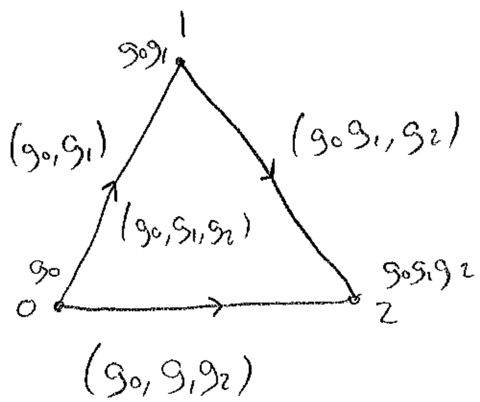
$$|\cdot| : (Set^G)^{\Delta^{op}} \longrightarrow Top^G$$

simplicial G-sets — G-spaces

In particular, EG is a G-space. Also I claimed last time that if you take $|\cdot|$ of $\overline{X}$ you get a space w/ one (contractible) component for each element of X. Since EG comes from $X = 1$, it will be contractible. In short, EG is a contractible G-space. It's universal in that if Z is any contractible G-space (e.g. a point)

$\exists! f : EG \to Z$ that's a map of G-spaces.

A typical 2-simplex in EG:



Note: · there's one vertex for each group element

· there's one edge from any vertex to any other:

$$g \xrightarrow{(g,g^{-1}h)} h$$

· there's one triangle for each triple of elements - these capture associativity:

above triangle "says" that $(g_0 g_1)g_2 = g_0(g_1 g_2)$

· one 3-simplex for each 4-tuple of elements, etc.

We can turn $\mathbb{E}G$ into a simplicial abelian group using the free functor from Set to AbGrp, which we'll call

$$\mathbb{Z}[-] : \mathrm{Set} \to \mathrm{AbGrp}$$
$$X \mapsto \mathbb{Z}[X] \text{ (not polynomials!)}$$

$\mathbb{Z}[X]$ consists of all $\mathbb{Z}$-linear combinations of elements of X.

$\mathbb{Z}[\mathbb{E}G]$ looks like

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathbb{Z}[G\times G\times G] \;\substack{\xrightarrow{\partial_0}\\\xrightarrow{\partial_1}\\\xrightarrow{\partial_2}}\; \mathbb{Z}[G\times G] \;\substack{\xrightarrow{\partial_0}\\\xrightarrow{\partial_1}}\; \mathbb{Z}[G]$$

with maps as before on generators.

We can then turn $\mathbb{Z}[G]$ into a chain complex of abelian groups:

$$\cdots \to \mathbb{Z}[G^3] \xrightarrow{\partial} \mathbb{Z}[G^2] \xrightarrow{\partial} \mathbb{Z}[G]$$

where
$$\underbrace{\partial_i(g_0,\dots,g_n)}_{\text{a generator for } \mathbb{Z}[G^{n+1}]} = \begin{cases} (g_0,\dots,g_i g_{i+1},\dots,g_n) & 0 \le i < n \\ (g_0,\dots,g_{n-1}) & i = n \end{cases}$$

and

$$\partial : \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$$

is $\partial = \sum_{i=0}^{n} (-1)^i \partial_i$

$\mathbb{E}G$ is a simplicial G-set, so $\mathbb{Z}[\mathbb{E}G]$ is a simplicial abelian G-module, ie a simplicial abelian group with an action of G, and our chain complex is a chain complex of G-modules and G-maps. Given $g \in G$ it acts on $\mathbb{Z}[G^n]$ as follows:

$$g(g_0, \ldots, g_{n-1}) = (gg_0, \ldots, g_{n-1})$$

and $\partial g(c) = g\partial(c) \quad \forall c \in \mathbb{Z}[G^n]$.

Lecture 15 - 11-2-2018

Group Cohomology

TRADITIONAL ALGEBRA	HOMOTOPY THEORY	HOMOLOGY THEORY

$$\begin{tikzcd}
\mathrm{Set}^G \arrow[r, "\text{bar construction}"] & (\mathrm{Set}^G)^{\Delta^{op}} \arrow[r, "\mathbb{Z}[-]"] \arrow[d, "\simeq"] & (\mathrm{AbGrp}^G)^{\Delta^{op}} \arrow[d, "\simeq"] \\
& (\mathrm{Set}^{\Delta^{op}})^G \arrow[d, "\text{geometric realization}"] & (\mathrm{AbGrp}^{\Delta^{op}})^G \arrow[d, "\text{chain complex}"] \\
& \mathrm{Top}^G & \mathrm{Ch}(\mathrm{AbGrp})^G \arrow[d] \\
& & (\mathrm{AbGrp}^G)^{\mathbb{N}}
\end{tikzcd}$$

$$(C^D)^E \simeq C^{D\times E} \simeq (C^E)^D$$

This is a standard pattern which people use for

- Set^G - group cohomology
- $R\mathrm{Mod}$ - cohomology of rings

etc.

Apply this machine to $*$, the 1-element set as a G-set w/ trivial action:

$$\begin{tikzcd}
* \arrow[r, mapsto] & \mathbb{E}G \arrow[r] \arrow[d, "\simeq"] & \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbb{Z}[G^2] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\simeq"] & \mathbb{Z}[G] \\
& \mathbb{E}G \arrow[d, "\simeq"] & C(G) = \arrow[r, "\partial"] & \mathbb{Z}[G^2] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\simeq"] & \mathbb{Z}[G] \\
& EG & & \mathbb{Z}[G^2] \arrow[r, "\partial"] & \mathbb{Z}[G]
\end{tikzcd}$$

Thrm - $H_n(C(G)) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$

Proof Sketch - This because we started with a point; EG is contractible so

$$\underbrace{H_n(EG)}_{\text{singular homology of top space}} = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$$

In general if $S \in Set^{\Delta^{op}}$ then the homology of its geometric realization $|S|$ is naturally isomorphic to the homology of $C(\mathbb{Z}[S])$ - the chain complex of the simplicial abelian group $\mathbb{Z}[S]$. ▨

We have a map of chain complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathbb{Z}[G^3] & \xrightarrow{\partial} & \mathbb{Z}[G^2] & \xrightarrow{\partial} & \mathbb{Z}[G] & \sum a_i g_i & a_i \in \mathbb{Z} \\ & & \downarrow & & \downarrow & & \downarrow & \downarrow & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \sum a_i & \end{array}$$

Which induces an isomorphism on homology using the theorem.

Also the $\mathbb{Z}[G^n]$ are free G-modules - free abelian groups on which G acts freely.

We call the top chain complex a <u>free resolution</u> of the bottom one - it's a puffed up version w/ same homology but everything free.

Def - Given any G-module A, let

$$C^n(G,A) = \hom_{AbGrp^G}(C_n(G), A)$$

($AbGrp^G$) is the category of G-modules), let

$$d: C^n(G,A) \longrightarrow C^{n+1}(G,A)$$

be

$$df(c) = f(\partial c)$$

Where $c \in C^{n+1}(G)$, $\partial c \in C^n(G)$ and $f \in C^n(G,A)$, so $f(\partial c) \in A$.

Def - The <u>group homology</u> of a group G with <u>coefficients</u> in a G-module A is:

$$H^n(G,A) = \frac{\ker(d: C^n(G,A) \to C^{n+1}(G,A))}{\operatorname{im}(d: C^{n-1}(G,A) \to C^n(G,A))}$$

Note $\operatorname{im} \subseteq \ker$ since $d^2 = 0$:

$$(d^2 f)(c) = (df)(\partial c) = f(\partial^2 c) = 0 \text{ since } \partial^2 = 0.$$

We'll see

$H^2(G,A)$ classifies short exact sequences

$$0 \longrightarrow A \xrightarrow{i} X \xrightarrow{p} G \longrightarrow 0$$

ways of glomming A and G together to form a bigger group.

$H^3(G,A)$ classifies ways of glomming A and G together to form a "2-group":

a 2-group is a category C w/ a multiplication:

$$m: C \times C \to C$$

and identity object $I \in C$ and inverses

$$\mathrm{inv}: C \to C$$

obeying group laws up to isomorphism.

$$\begin{array}{lll} & \pi_1 & G \\ & \pi_2 & A \\ \text{abelian} & \pi_3 & 0 \\ & \vdots & 0 \\ & & \vdots \end{array}$$

Lecture 16 11-5-2018

Extensions of Groups

The Fundamental Thrm of Arithmetic says any (nonzero) natural number is a product of primes, uniquely up to reordering. Something similar is true for finite groups where the "atoms" are the simple groups: groups w/o any nontrivial normal subgroups. For finite abelian groups, the simple ones are $\mathbb{Z}_p$ with p prime, but not every finite abelian group is a product of $\mathbb{Z}_p$.

For example, the 4 element abelian groups are $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$, which is built from two $\mathbb{Z}_2$'s as an "extension": it has $\mathbb{Z}_2$ as a normal subgroup, the quotient is $\mathbb{Z}_2$.

Jordan-Holder Thrm - If G is any finite group, there is a <u>composition series</u> for G: subgroups

$$1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G$$

such that

1) each H_i is normal in H_{i+1} $(0 \leq i < n)$
2) H_{i+1}/H_i is simple

The composition series is not unique but the simple groups H_{i+1}/H_i $(0 \leq i < n)$ are unique up to reordering

Def - We say a group E is an extension of the group G by N if N is a normal subgroup of E and $E/N = G$, i.e.

$$1 \to N \to E \to G \to 1$$

is exact.

So to completely classify finite groups, we "just" need to classify finite simple groups (proof: $\sim$ 10,000 pages) and then understand extensions. We'll just study <u>abelian extensions</u>:

$$1 \to A \to E \to G \to 1$$

where A is abelian. These are classified by $H^2(G, A)$.
For nonabelian extensions we'd need "nonabelian cohomology" - which uses even more 2-category theory.

Suppose

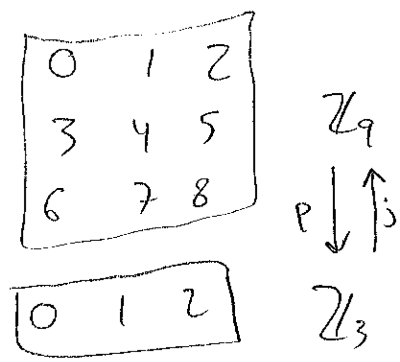
$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

is an abelian extension; we can think of A as a subset of E so we write $i(a) = a$.
We can choose a function $j : G \to E$:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, hook] & E \arrow[r, bend left, "p"] & G \arrow[l, bend left, "j"] \arrow[r] & 1 \end{tikzcd}$$

such that $p \circ j = 1_G$. E.g.

$$1 \to \mathbb{Z}_3 \hookrightarrow \mathbb{Z}_9 \to \mathbb{Z}_3 \to 1$$



Since elements of G are just cosets Ae where $e \in E$ we can always choose $j(g)$ to be any element of the coset that is g, and then $p(j(g)) = g$. Typically j is not a homomorphism!
Then each element $e \in E$ is of the form $aj(g)$ for some unique $g \in G$ and $a \in A$: g says which coset e is in, while a says where e is in the coset.

So we get a bijection

$$A \times G \xrightarrow{\sim} E$$
$$(a,g) \longmapsto aj(g)$$

How does multiplication in E work, in terms of these pairs?

$$aj(g)\,a'j(g') \qquad\qquad g,g' \in G$$
$$\qquad\qquad\qquad a,a' \in A$$

$$\| $$

$$a\underbrace{j(g)a'j(g)^{-1}}_{\in A \text{ since } A \text{ normal}}\,j(g)j(g')$$

$$\|$$

$$a\underbrace{j(g)a'j(g)^{-1}}_{\substack{\text{call this} \\ \alpha(g)a'}}\underbrace{j(g)j(g')j(gg')^{-1}}_{\substack{\text{this is in } A \text{ too!} \\ \text{call it } c(g,g')}}j(gg')$$

Thm - α is an action of G on A making A into a G-module. Namely:

$$\alpha(g)(aa') = (\alpha(g)(a))(\alpha(g)(a'))$$
$$\alpha(gg')(a) = \alpha(g)(\alpha(g')(a))$$

$c: G \times G \longrightarrow A$ and c obeys the <u>2-cocycle condition</u>:

$$\alpha(g)c(g',g'') - c(gg',g'') + c(g,g'g'') - c(g,g') = 0$$

where now we write the group operation in A using $+$.

Proof - HW, due Friday.

The 2-cocycle condition comes from the associative law for E. ☐

This is how group cohomology was discovered. The connection to simplices involves the tetrahedron. In G, the assoc law says:

$$
\begin{tikzcd}
\bullet \arrow[r, "g'"] & \bullet \arrow[d, "g''"] \\
\bullet \arrow[u, "g"] \arrow[ur, "gg'"'] \arrow[r, "(gg')g''"'] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "g'"] \arrow[dr, "g'g''"] & \bullet \arrow[d, "g''"] \\
\bullet \arrow[u, "g"] \arrow[r, "g(g'g'')"'] & \bullet
\end{tikzcd}
$$

In $E \cong A \times G$ we have

$(a,g)(a',g') = (a + \alpha(g)a' + c(g,g'), gg')$

and associativity of this is connected to the 2-cocycle condition.

$$
\begin{tikzcd}
\bullet \arrow[r, "g'"] & \bullet \arrow[d, "g''"] \\
\bullet \arrow[u, "g"] \arrow[ur, "gg'"'] \arrow[ul, phantom, "\Downarrow c(g,g')"] \arrow[r, phantom, "\Downarrow c(gg',g'')"'] & \bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[r, "g'"] \arrow[dr, "g'g''"] \arrow[r, phantom, "\Downarrow c(g',g'')"', shift right=3] & \bullet \arrow[d, "g''"] \\
\bullet \arrow[u, "g"] \arrow[r, phantom, "\Downarrow c(g,g'g'')"] & \bullet
\end{tikzcd}
$$

Lecture 17 11-7-2018

Extensions of Groups

Last time we almost showed:

Thm - Given a group $(G, \cdot, 1)$ and an abelian group $(A, +, 0)$ there's a short exact sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$$

where $E = A \times G$ with multiplication given by

$$(a, g)(a', g') = (a + \alpha(g)a' + c(g, g'), gg')$$

with $i(a) = (a, 1)$

$p(a, g) = g$

where:

* 1) $\alpha : G \to \mathrm{Aut}(A)$, so
A is a G-module

* 2) $C : G \times G \to A$ is a 2-cocycle:

$$\alpha(g) c(g', g'') - c(gg', g'') + c(g, g'g'') - c(g, g')$$

Multiplication in E is assoc

3) c is normalized: conditions coming from E has an identity (0, 1) and inverses, including:

$$c(1, g) = c(g, 1) = 0 \quad \forall g \in G$$

Moreover, every extension of G by A: $1 \to A \to E \to G \to 1$ is isomorphic to one of this form.

Proof- In HW, took any extension of G by A and showed it had this form (or at least everything except 3)

Converse is just a calculation. ∎

There are four cases:

	$\alpha(g)=1_A$	perhaps α is nontrivial
$c(g,g')=0$	Direct Product $A \times G$	Semidirect product $A \ltimes G$ (A normal subgroup of G)
perhaps c is nonzero	Central extension of G by A	Extension of G by A

If $c=0$:

$$(a,g)(a',g')=(a+\alpha(g)a',\,gg')$$

If $\alpha=1$:

$$(a,g)(a',g')=(a+a'+c(g,g'),\,gg')$$

This happens iff A is a subgroup of the center of E. Recall:

$\alpha(g)a = j(g)\,a\,j(g)^{-1}$ given

$$1 \longrightarrow A \xrightarrow{\;i\;} E \underset{j}{\overset{p}{\rightleftarrows}} G \longrightarrow 1$$

So if α trivial, $a \in A$ commutes w/ all $(0,g)=j(g)\in E$, and since A is abelian, a also commute w/ everyone in A.

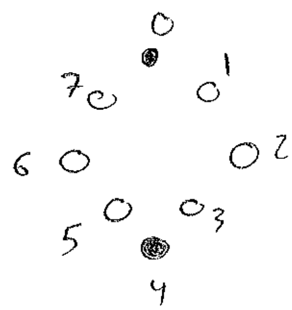
Consider 8-element groups,
None simple, so all are built by extensions:

1) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is a direct product

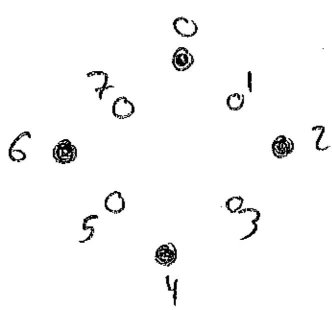
2) $\mathbb{Z}_2 \times \mathbb{Z}_4$ is a direct product

3) The dihedral group D_8 is a semidirect product $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$
reflections of the square act on rotations.

4) $\mathbb{Z}_8$ is a central extension of $\mathbb{Z}_4$ by $\mathbb{Z}_2$:



Also it's a central extension of $\mathbb{Z}_2$ by $\mathbb{Z}_4$:



5) The quaternion group:

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

relations including $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$ and cyclic permutations

$$1 \to \underset{\mathbb{Z}_2}{\underset{\parallel}{\{\pm 1\}}} \to Q_8 \to \underset{\mathbb{Z}_2 \times \mathbb{Z}_2}{\underset{\parallel}{\{1, i, j, k : i^2 = j^2 = k^2 = 1\}}} \to 1$$

This is a central extension.

Of the ~50 billion groups of order < 2000, over 99% have order $1024 = 2^{10}$. "Most" finite groups have order that's a power of 2 - they're built from $\mathbb{Z}_2$ using iterated extensions.

Lecture 18 11-9-2018

A 2-cocycle in everyday

An odometer describes $\mathbb{Z}_{1,000,000}$ as an iterated central extension involving 6 copies of $\mathbb{Z}_{10}$. Consider $\mathbb{Z}_{100}$:

$$\begin{array}{r} {}^{1}19 \\ +23 \\ \hline 42 \end{array}$$

We're writing $\mathbb{Z}_{100}$ as $\mathbb{Z}_{10} \times \mathbb{Z}_{10}$ with addition:

$$(a,b)+(a',b') = (a+a' + c(b,b'), b+b')$$

where $c(b,b')$ is the carry digit. Associativity gives c the cocycle property.

Group cohomology

Using the bar construction, we turned the one-point G-set $*$ into a chain complex of G-modules:

$$\mathrm{Set}^G \xrightarrow{} (\mathrm{Set}^G)^{\Delta^{op}} \to (\mathrm{AbGrp})^{\Delta^{op}} \xrightarrow{c} \mathrm{Ch}(\mathrm{AbGrp}^G)$$

$$* \longmapsto \cdots \to \mathbb{Z}[G^3] \xrightarrow{\partial} \mathbb{Z}[G^2] \xrightarrow{\partial} \mathbb{Z}[G]$$

The cohomology of G w/ coefficients in the G-module A is cohomology of the cochain complex

$$\hom_{(\mathrm{AbGrp})^G}(\mathbb{Z}[G^n], A)$$

with d that increases n by one:

$$df(c) = f(\partial c)$$

Let's simplify this! There's an isomorphism

$$\beta : \hom_{(AbGrp)^G}(\mathbb{Z}[G^{n+1}], A) \xrightarrow{\sim} \hom_{AbGrp}(\mathbb{Z}[G^n], A)$$

$$f \longmapsto \tilde{f}$$

Where

$$\tilde{f}(g_1, g_2, \ldots, g_n) = f(1, g_1, \ldots, g_n)$$

Note:
$$\begin{aligned} f(g_0, g_1, \ldots, g_n) &= f(g_0(1, g_1, \ldots, g_n)) \\ &= \alpha(g_0)(f(1, g_1, \ldots, g_n)) \\ &= \alpha(g_0)\tilde{f}(g_1, \ldots, g_n) \end{aligned}$$

So we can recover f from $\tilde{f}$, so β is an isomorphism.

So we can define $\tilde{d}$ uniquely so that

$$\begin{tikzcd} \hom_{(AbGrp)^G}(\mathbb{Z}[G^n], A) \arrow[r, "\beta", "\sim"'] \arrow[d, "d"'] & \hom_{AbGrp}(\mathbb{Z}[G^{n-1}], A) \arrow[d, "\tilde{d}"] \\ \hom_{(AbGrp)^G}(\mathbb{Z}[G^{n+1}], A) \arrow[r, "\beta", "\sim"'] & \hom_{AbGrp}(\mathbb{Z}[G^n], A) \ni \tilde{d}\tilde{f} \end{tikzcd}$$

commutes. $\tilde{d} = \beta \circ d \circ \beta^{-1}$, or

$$\tilde{d}\tilde{f} = \widetilde{df}$$

Work it out!

$$\tilde{d}\tilde{f}(g_1,\dots,g_n) = \widetilde{df}(g_1,\dots,g_n)$$
$$= df(1,g_1,\dots,g_n)$$
$$= f\partial(1,g_1,\dots,g_n)$$
$$= f(g_1,\dots,g_n) + \sum_{i=1}^{n-1}(-1)^i f(1,g_1,\dots,g_ig_{i+1},\dots,g_n) + (-1)^n f(1,g_1,\dots,g_{n-1})$$

$$= \alpha(g_1)\tilde{f}(g_2,\dots,g_n) + \sum_{i=1}^{n-1}(-1)^i \tilde{f}(g_1,\dots,g_ig_{i+1},\dots,g_n) + (-1)^n\tilde{f}(g_1,\dots,g_{n-1})$$

Example: an extension of G by A gives a map $c: G^2 \to A$;

this is a 2-cocycle, ie $\tilde{d}c = 0$, iff

$$\alpha(g_1)c(g_2,g_3) - c(g_1g_2,g_3) + c(g_1,g_2g_3) - c(g_1,g_2) = 0$$

This is exactly the condition needed for our extension built using α and c to be associative!

Thrm - Extensions of a group G by an abelian group A are classified up to isomorphism by:

- an action α of G on A
- a cohomology class
$$[c] \in H^2(G,A) = \frac{\{2\text{-cocycles}\}}{\{2\text{-coboundaries}\}}$$

Idea of proof - Starting from an extension

$$1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$$

and a map $j: G \to E$ w/ $p \circ j = 1_G$, we got an action α and a 2-cocycle c.

If we changed j, we'd get the same α and c would change to $c' = c + dk$ for some $k : G \to A$.

Conversely, given an action α and a 2-cocycle c, we can build a group structure on $E = A \times G$ and an extension

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

If we choose c' with $[c] = [c']$, then $c' = c + dk$, we get another extension but it's isomorphic:

$$\begin{tikzcd} & & E \arrow[dd, "\ell", "\wr"'] \arrow[dr] & & \\ 1 \arrow[r] & A \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

i.e. there's an isomorphism $\ell : E \to E'$ that makes this commute. ▨

Lecture 19 11-16-2018

No more class except: Fridays (except day after Thanksgiving)
Monday Nov 26^{th}

But classes are optionally extended for a 2nd hour

Examples of Group Cohomology

Suppose p and q are prime.

Prop - Let $\mathbb{Z}_q$ be made into a $\mathbb{Z}_p$-module in a trivial way ($\mathbb{Z}_p$ acts as identity automorphisms of $\mathbb{Z}_q$)

$$H^2(\mathbb{Z}_p, \mathbb{Z}_q) = \begin{cases} \mathbb{Z}_p & \text{if } p = q \\ \{0\} & \text{if } p \neq q \end{cases}$$

So the only central extension of $\mathbb{Z}_p$ by $\mathbb{Z}_q$ is the trivial one:

$$0 \to \mathbb{Z}_q \to \mathbb{Z}_q \oplus \mathbb{Z}_p \to \mathbb{Z}_p \to 0$$

But when $p=q$ we get p nonisomorphic central extensions:

$$0 \to \mathbb{Z}_p \to E \to \mathbb{Z}_p \to 0$$

But in all of these, E is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$

Note if $p \neq q$, you do have

$$0 \to \mathbb{Z}_q \to \mathbb{Z}_{pq} \to \mathbb{Z}_p \to 0$$

but $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$

Much more generally:

Fracture Theorem:

1) Any group of order p^n is <u>nilpotent</u>:

a finite group is nilpotent iff it's built from abelian groups by iterated central extensions:

$$1 \to A_1 \to E_1 \to A_0 \to 1$$

$$1 \to A_2 \to E_2 \to A_1 \to 1$$

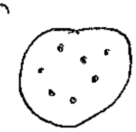
with our group being E_n and all A_i abelian.

2) Any group of order p^n is built from $\mathbb{Z}_p$ by iterated central extensions. (Just show $\mathbb{Z}_{p^n}$ is a central extension of $\mathbb{Z}_{p^{n-1}}$ by $\mathbb{Z}_p$, so all abelian groups of order a power of p are built from $\mathbb{Z}_p$ by iterated central extensions.)

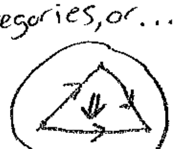
3) Any finite nilpotent group is a direct product of groups (necessarily nilpotent) of prime power order for various primes.

Higher-dimensional algebra or "Homotopification"

We can generalize familiar algebraic structures (groups, rings, ...) from sets to higher dimensional things such as:



- categories (or 2-categories, or...)
- simplicial sets
- topological spaces



These are related! Let Cat be the mere category of all (small) categories. Then we have functors:

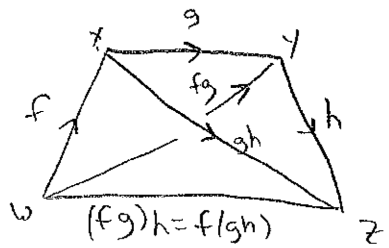
$$\mathrm{Cat} \xrightarrow[\text{"nerve"}]{N} \mathrm{Set}^{\Delta^{op}} \xrightarrow[\text{"geometric realization"}]{|\cdot|} \mathrm{Top}$$

The nerve of a category C has 0-simplices

$\bullet\, x$ objects of C

$x \xrightarrow{f} y$ morphisms of C

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "gf"'] & & z \end{tikzcd}$$ commuting triangles in C



commuting tetrahedra in C

Def - Given a category C it's <u>nerve</u> $N(C): \Delta^{op} \to Set$ is given by:

$$N(C)([n]) = \hom_{Cat}([n], C)$$

Where $[n]$ is the totally ordered set $\{0, 1, \ldots, n-1\}$, which is a poset hence a category:

$$\underset{0}{\bullet} \to \underset{1}{\bullet} \to \underset{2}{\bullet} \cdots\cdots \to \underset{n-1}{\bullet}$$

For example if $[3] = \{0,1,2\}$

$N(C)([3]) = \{$ functors $F: [3] \to C\}$

$$= \left\{ \begin{tikzcd} & F(1) \arrow[dr] & \\ F(0) \arrow[ur] \arrow[rr] & & F(2) \end{tikzcd} \right\}$$

Thm - 1) Every CW complex is homotopy equivalent to $|X|$ for some $X \in Set^{\Delta^{op}}$

2) Every space $|X|$ with $X \in Set^{\Delta^{op}}$ is homeomorphic to $|N(C)|$ from some category C.

Also, geometric realization

$$\begin{tikzcd} \Delta \arrow[r, hook, "Y"] \arrow[d] & Set^{\Delta^{op}} \arrow[dl, dotted, "|\cdot|"] \\ Top & \end{tikzcd}$$

$[2] \to [3]$

has a right adjoint

$Top \to Set^{\Delta^{op}}$

also called "nerve".

Extra hour for Lecture 19 11-16-2018

What's this other nerve

$$\mathcal{N} : \mathrm{Top} \to \mathrm{Set}^{\Delta^{op}} ?$$

$$\mathcal{N}(X) : \Delta^{op} \to \mathrm{Set}$$

is given by

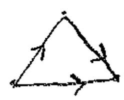
y yoneda embedding

$|\cdot|$ geometric realization

$$\mathcal{N}(X)([n]) = \hom(|y[n]|, X)$$

where

$$\begin{tikzcd} \Delta \arrow[r, hook, "Y"] \arrow[d] & \mathrm{Set}^{\Delta^{op}} \arrow[dl, "|\cdot|"] \\ \mathrm{Top} & \end{tikzcd}$$



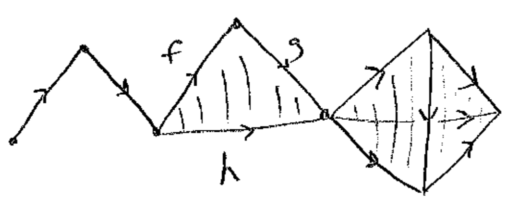
In algebraic topology we use this to turn a space into a chain complex:

$$\mathrm{Top} \xrightarrow{\mathcal{N}} \mathrm{Set}^{\Delta^{op}} \xrightarrow{\mathbb{Z}[\,]} \mathrm{AbGrp}^{\Delta^{op}} \xrightarrow{C} \mathrm{Ch}(\mathrm{AbGrp})$$

So we have

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend left, "N"] & \mathrm{Set}^{\Delta^{op}} \arrow[l, bend left] \arrow[r, bend left, "|\cdot|"] & \mathrm{Top} \arrow[l, bend left, "\mathcal{N}"] \end{tikzcd}$$

where ? turns any simplicial set into a category - left adjoint to N:



where $gf = h$

The theorem I stated is a step toward proving that Cat, $\mathrm{Set}^{\Delta^{op}}$ and Top are "Quillen equivalent model categories" - they're all equally good for homotopy theory, even though they're not equivalent as categories

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$\mathbb{E}G$, revisited

Given a group G, we got a simplicial set $\mathbb{E}G$

$$\begin{tikzcd}
 & g_0g_1 \arrow[dr, "{(g_0g_1,\, g_2)}"] & \\
g_0 \arrow[ur, "{(g_0,g_1)}"] \arrow[rr, "{(g_0,\, g_1g_2)}"'] & {(g_0,g_1,g_2)} & g_0g_1g_2
\end{tikzcd}$$

One 0-simplex for each group element and one 1-simplex for each pair (g,h):

$$g \overset{(g,h)}{\text{———}} gh$$

This is the nerve of a category!

This category has:

- elements of G as objects
- there's exactly one morphism from any $g \in G$ to any $k \in G$

$$g \xrightarrow{(g,h)} k \qquad \text{where } h \text{ is such that } gh = k.$$

This is a groupoid, and an example of this:

Def - If X is a set with a right G-action, the <u>translation groupoid</u> $X//G$ has:

- one object for each $x \in X$
- morphisms like this:

$$x \xrightarrow{(x,g)} y \quad \text{such that} \quad xg = y, \quad x, y \in X$$

- composition works like this:

$$\begin{tikzcd} & xg \arrow[dr, "{(xg,h)}"] & \\ x \arrow[ur, "{(x,g)}"] \arrow[rr, "{(x,gh)}"'] & & xgh \end{tikzcd}$$

Theorem - $N(G//G) = \mathbb{E}G$, where G acts on itself by right multiplication.

Proof - Just look at them! $\boxed{/\!/}$

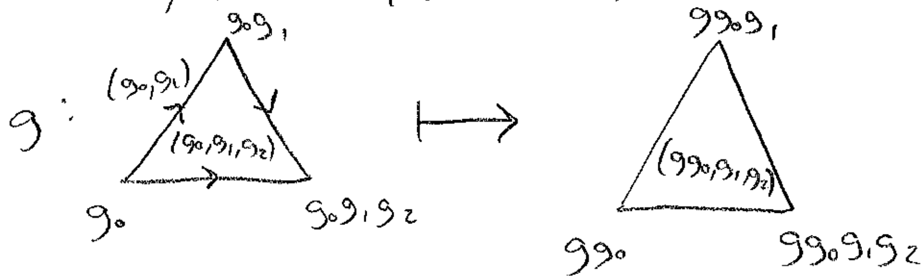
In general, $X//G$ is the "weak quotient" of X by G where we decree $x, y \in X$ are isomorphic (not equal) if $\exists g \in G$ such that $xg = y$.

There's a map $\underbrace{X//G}_{\text{groupoid}} \to \underbrace{X/G}_{\text{a set, hence a groupoid w/ only identity elements}}$

$$x \xrightarrow{(x,g)} g \quad \longmapsto \quad [x] \xrightarrow{id_{[x]}} [xg] = [x]$$

Which turns isomorphisms into equations.

G/G is just a point, while $|N(G/G)| = EG$ is a contractible space. G acts trivially on a point, but G acts nontrivially on the left of $G//G$, hence $N(G//G)$, hence $|N(G//G)|$:



So $\mathbb{E}G$ and EG have nontrivial left G-actions.

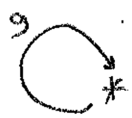
Thrm - Let $BG = EG/G$ where we mod out by the left G-action. Then

1) BG is connected,

   $\pi_1(BG) \cong G$

   $\pi_n(BG) = \{0\}$ $n > 1$

2) Any CW complex with these properties is homotopy equivalent to BG.



$g \in G$ gives an element of $\pi_1(BG)$

Lecture 20 11-26-2018

Homotopification and Higher Algebra

We can generalize familiar mathematical structures from the category of sets to categories of "higher-dimensional" things:

e.g.

$$\mathrm{Set} \xrightarrow{\mathrm{Disc}} \mathrm{Cat} \xrightarrow[\text{Nerve}]{N} \mathrm{Set}^{\Delta^{op}} \xrightarrow[\text{Geo. Realization}]{|\cdot|} \mathrm{Top}$$

"0d math" "1d math" "higher-dim math"

All these categories have products, which enable us to define:

monoids	groups	rings
commutative monoids	abelian groups	commutative rings

inside any of these categories.

A monoid in Cat, for example, is called a "strict monoidal category".

Def - A category X has <u>products</u> if given any family of objects $\{X_\alpha\}_{\alpha \in A}$ in X there's an object Y called their product, equipped w/ projections $\pi_\alpha : Y \to X_\alpha$ such that given any other morphisms $f_\alpha : q \to X_\alpha$, there exists a unique morphism $g : q \to Y$ such that:

$$\begin{tikzcd} & q \arrow[dl, "g"'] \arrow[d, "f_\alpha"] \\ Y \arrow[r, "\pi_\alpha"'] & X_\alpha \end{tikzcd}$$

Example: If $A = \emptyset$, we're saying for any object q, $\exists! g : q \to Y$, and we say Y is a terminal object, usually denoted 1. E.g. in Cat,

$1 = x \circlearrowleft 1_x$

and in $\mathrm{Set}^{\Delta^{op}}$ the terminal object has one n-simplex for each $n \in \mathbb{N}$. In Top, 1 is the singleton w/ only topology.

Def - If X has products, a <u>monoid in X</u>, or <u>monoid internal to X</u>, or <u>monoid internal to X</u>, or <u>monoid object in X</u>, is an object $M \in X$ with a multiplication:

$$u : M \times M \longrightarrow M$$

and unit $i : 1 \to M$

obeying <u>associativity</u>

$$\begin{tikzcd} M \times M \times M \arrow[r, "u \times 1"] \arrow[d, "1 \times u"'] & M \times M \arrow[d, "u"] \\ M \times M \arrow[r, "u"] & M \end{tikzcd}$$

and left/right unit laws

$$\begin{tikzcd} 1 \times M \arrow[d, "i \times 1"'] & M \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "1_M"] & M \times 1 \arrow[d, "1 \times i"] \\ M \times M \arrow[r, "u"] & M & M \times M \arrow[l, "u"'] \end{tikzcd}$$

We can also define morphisms of monoids in X, which are morphisms $f : M \to M'$ that preserve u and i.

Puzzle: What commutative diagrams express this "preservation" of u and i?

There's a category $\mathrm{Mon}(X)$ of monoids in X.

For example:

- a monoid in Set is a <u>monoid</u>: $\mathrm{Mon}(\mathrm{Set}) = \mathrm{Mon}$
- a monoid in Cat is a <u>strict monoidal category</u>: there's a category $\mathrm{Mon}(\mathrm{Cat})$
- a monoid in $\mathrm{Set}^{\Delta^{op}}$ is called a <u>simplicial monoid</u>: there's a category $\mathrm{Mon}(\mathrm{Set}^{\Delta^{op}}) \simeq \mathrm{Mon}^{\Delta^{op}}$
- a monoid in Top is called a <u>topological monoid</u>, e.g. $([0,\infty), +, 0) \in \mathrm{Mon}(\mathrm{Top})$.

These functors preserve products:

$$\mathrm{Set} \xrightarrow{\mathrm{Disc}} \mathrm{Cat} \xrightarrow{N} \mathrm{Set}^{\Delta^{op}} \xrightarrow{|\cdot|} \mathrm{Top}$$

e.g.

$$N\Big(\prod_{\alpha \in A} C_\alpha\Big) \cong \prod_{\alpha \in A} N(C_\alpha)$$

is a canonical isomorphism.

If $F : X \to X'$ preserves products and these categories have products, we get a functor

$$\mathrm{Mon}(F) : \mathrm{Mon}(X) \longrightarrow \mathrm{Mon}(X')$$

So we can turn monoids into strict monoidal categories, and then into simplicial monoids, and then into topological monoids.

$\overset{\bullet}{0} \quad \overset{\bullet}{1} \longmapsto \overset{\bullet}{0} \quad \overset{\bullet}{1}$ Thm - Right adjoints preserve products

Let's prove this for binary products (same argument works in general for any product. Say X and X' have binary products and $U: X \to X'$ is a right adjoint, want $U(a \times b) \cong U(a) \times U(b)$ for all $a, b \in X$.

Let's show $U(a \times b)$ is a product of $U(a)$ and $U(b)$ that will do it. We need projections

$$\begin{tikzcd} & U(a\times b) \arrow[dl] \arrow[dr] & \\ U(a) & & U(b) \end{tikzcd}$$

We had

$$\begin{tikzcd} & a\times b \arrow[dl, "\pi_a"'] \arrow[dr, "\pi_b"] & \\ a & & b \end{tikzcd}$$

so we set our projections from that

$$\begin{tikzcd} & U(a\times b) \arrow[dl, "U(\pi_a)"'] \arrow[dr, "U(\pi_b)"] & \\ U(a) & & U(b) \end{tikzcd}$$

Given any competitor

$$\begin{tikzcd} & Q \arrow[d, dashed, "g"] \arrow[ddl, bend right, "f_a"'] \arrow[ddr, bend left, "f_b"] & \\ & U(a\times b) \arrow[dl, "U(\pi_a)"'] \arrow[dr, "U(\pi_b)"] & \\ U(a) & & U(b) \end{tikzcd}$$

☆

Now recall $\hom_{X'}(Fx, x') \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \hom_X(x, Ux')$ for all objects $x \in X$, $x' \in X'$ where F is the left adjoint.

In X we have

$$\begin{tikzcd} & F(Q) \arrow[ddl, bend right, "\beta(f_a)"'] \arrow[d, dashed, "\exists! h"] \arrow[ddr, bend left, "\beta(f_b)"] & \\ & a \times b \arrow[dl, "\pi_a"'] \arrow[dr, "\pi_b"] & \\ a & & b \end{tikzcd}$$

☆ ☆

since $a \times b$ is a product. Take $g = \alpha(h)$. Show ☆ commutes for this g iff ☆ ☆ commutes for the corresponding h.

Which of these are right adjoints?

$$\mathrm{Set} \xrightarrow{\mathrm{Disc}} \mathrm{Cat} \xrightarrow{N} \mathrm{Set}^{\Delta^{op}} \xrightarrow{|\cdot|} \mathrm{Top}$$

$|\cdot|$ is a left adjoint

$$\begin{tikzcd} \mathrm{Top} \arrow[r, bend left, "n \text{ - is a right adjoint}"] & \mathrm{Set}^{\Delta^{op}} \arrow[l, bend left, "|\cdot| \text{ - is a left adjoint}"] \end{tikzcd}$$

N is a right adjoint:

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend left, "N"] & \mathrm{Set}^{\Delta^{op}} \arrow[l, bend left, "\text{realization}"] \end{tikzcd}$$

There's a forgetful functor $U: \mathrm{Cat} \to \mathrm{Set}$ that only remembers the set of objects. This has a left adjoint $L: \mathrm{Set} \to \mathrm{Cat}$ and right adjoint $R: \mathrm{Set} \to \mathrm{Cat}$.

Which one is $\mathrm{Disc}: \mathrm{Set} \to \mathrm{Cat}$?

$$\hom_{\mathrm{Cat}}(\mathrm{Disc}(S), C) \cong \hom_{\mathrm{Set}}(S, UC)$$

so $\mathrm{Disc} = L$, the left adjoint

So N preserves products because it's a right adjoint, but Disc and $|\cdot|$ do so for subtler reasons.

Products in Cat look like this:

C

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$C' \quad \begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad C \times C'$$

where the triangles commute.

So if $S = \{\bullet \ \bullet\}$, $S' = \{\vdots\}$

$$S \times S' = \begin{array}{c|cc} & & S \\ & \bullet & \bullet \\ S' & \bullet & \bullet \\ & \bullet & \bullet \end{array}$$

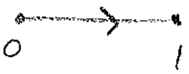
$$\mathrm{Disc}(S \times S') = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array} \cong \mathrm{Disc}(S) \times \mathrm{Disc}(S')$$

$|\cdot|$ preserves products for similar reasons.

Given $X, X' \in Set^{\Delta^{op}}$,

$$(X \times X')[n] = X[n] \times X'[n]$$

Suppose X and X' are the walking 1-simplex



or really

$X[n] = hom_\Delta([n], [2])$

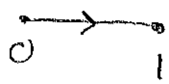
since

$[2] = \{0,1\}$

So 0-simplices in the walking 1-simplex are

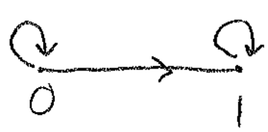
$$hom_\Delta([1],[2]) = hom_\Delta(\{0\}, \{0,1\})$$

↑
namely
finite ordinals

There are 2: $0 \mapsto 0$, $0 \mapsto 1$ shown here 

The 1-simplices in X are $hom_\Delta(\{0,1\},\{0,1\})$ are:

$0 \mapsto 0$	$0 \mapsto 0$	$0 \mapsto 1$
$1 \mapsto 0$	$1 \mapsto 1$	$1 \mapsto 1$

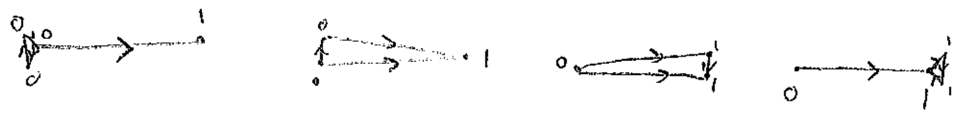


Two of these are "degenerate".

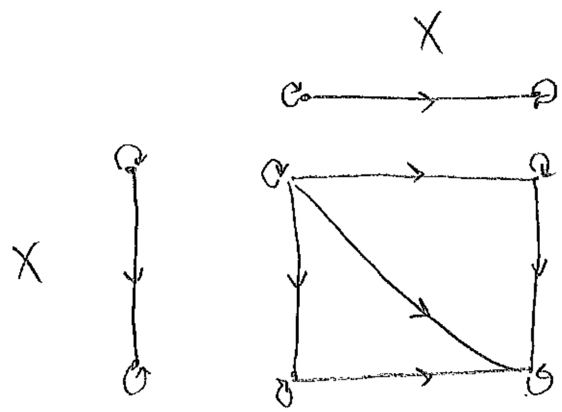
The 2-simplices in X are

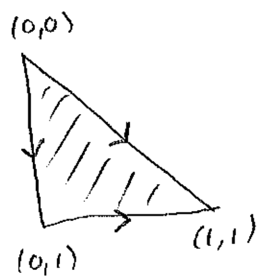
$\hom_\Delta(\{0,1,2\},\{0,1\})$

$$\begin{array}{ccc}0&1&2\\ \downarrow&\downarrow&\downarrow\\ 0&0&0\end{array}\qquad\begin{array}{ccc}0&1&2\\ \downarrow&\downarrow&\downarrow\\ 0&0&1\end{array}\qquad\begin{array}{ccc}0&1&2\\ \downarrow&\downarrow&\downarrow\\ 0&1&1\end{array}\qquad\begin{array}{ccc}0&1&2\\ \downarrow&\downarrow&\downarrow\\ 1&1&1\end{array}$$

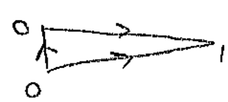


These are needed for $|\cdot|$ to preserve products:

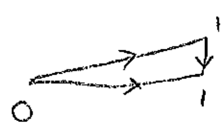




comes from product of 2 degenerate triangles



and



Two degenerate triangles $\Rightarrow$ nondegenerate triangle

Lecture 21 11-30-2018

Monoidal Categories

We defined a <u>strict monoidal category</u> to be a monoid in Cat, ie a category M w/ a functor usually called

$$\otimes : M \times M \longrightarrow M$$

and a functor

$$i : 1 \longrightarrow M \quad (\text{the same as an object } I \in M)$$

obeying associativity and left/right unit laws:

$$I \otimes x = x = x \otimes I \quad \forall x \in M$$

Thrm- A strict monoidal category is "the same" as a 2-category with one object.

Proof- Given a 2-category C with one object $* \in C$, let $M = hom_C(*,*)$, let $\otimes = \circ : hom_C(*,*) \times hom_C(*,*)$ and let $I = 1_* \in hom_C(*,*)$. The converse works the same way. ▨

This is a categorification of the fact that a monoid is a category with one object.

The Strict Periodic Table

Sets | Categories | (strict) 2-categories | (strict) 3-categories

monoids = Cats w/ 1 object

Strict mon Cats = 2-cats w/ 1-object

Strict mon 2-cats = strict 3-cats w/ 1 object

comm monoids = 2 cats w/ 1 object and 1 morphism = strict mon Cats w/ 1 object

Commutative strict monoidal cats = strict 2-cats w/ 1 object = 3 cats w/ 1 object and 1 morphism

comm monoids = strict 3-cats w/ 1 object, 1 morphism, 1 2-morphism

Here, we are reindexing, e.g.

2-cat with one object	strict mon cat
2-morphisms	morphisms
morphisms	objects
*	

Showing that, mon cats w/ one object are comm. monoids is the "Eckmann-Hilton argument", invented by topology to show $\pi_2(X,*)$ is an abelian group.

$\pi_0(X,*)$ = the <u>set</u> are connected components of X

$\pi_1(X,*)$ = the <u>group</u> of homotopy classes of pointed maps $S^1 \to X$

$\pi_2(X,*)$ = the <u>abelian group</u> of...

$\pi_3(X,*)$ = the abelian group of...

This secretly comes from the 1st column in the periodic table.

Most categories with a tensor product are not strict monoidal categories: instead we just have natural isomorphisms:

$\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$ the associator

$\lambda_x : I \otimes x \xrightarrow{\sim} x$ left unitor

$\rho_x : x \otimes I \xrightarrow{\sim} x$ right unitor

In a monoidal category we have a cat M w/ a functor $\otimes : M \times M \to M$ and an object $I \in M$ and these 3 natural isomorphisms obeying some "coherence laws" which mean we don't have to care about <u>how</u> we reparanthesize:

$$\begin{tikzcd}
& (w\otimes(x\otimes y))\otimes z \arrow[r, "\alpha_{w,x\otimes y,z}"] & w\otimes((x\otimes y)\otimes z) \arrow[d, "1_w \otimes \alpha_{x,y,z}"] \\
((w\otimes x)\otimes y)\otimes z \arrow[ur, "\alpha_{w,x,y}\otimes 1_z"] \arrow[dr, "\alpha_{w\otimes x,y,z}"'] & & \\
& (w\otimes x)\otimes(y\otimes z) \arrow[r, "\alpha_{w,x,y\otimes z}"'] & w\otimes(x\otimes(y\otimes z))
\end{tikzcd}$$

commutes (the <u>pentagon identity</u>), and also...

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

commutes (the <u>triangle identity</u>). MacLane and Kelly showed around 1968 that these imply that every diagram built from associators, unitors, tensoring and identities commutes.

Replacing equations w/ isomorphisms obeying coherence laws is called <u>weakening</u>.

There are lots of non-strict monoidal cats:

$(Set, \times, 1)$

$(Set, +, \emptyset)$

$(Vect_k, \otimes, k)$

$(Vect_k, \oplus, \{0\})$

Lecture 22 12-7-2018

n-Groupoids and The Periodic Table

Using general (non-strict) monoidal categories, we can define a nice notion of categorified group or "2-group".

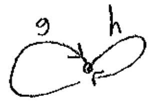
Def - A 2-group M is a monoidal category where:

1) Every morphism is invertible: given $f: x \to y$, there exists $f^{-1}: y \to x$ such that $f \circ f^{-1} = 1_y$, $f^{-1} \circ f = 1_x$ (ie M is a groupoid)

2) Every object $g \in M$ is invertible: $\exists g^{-1} \in M$ such that $g \otimes g^{-1} \cong I$, $g^{-1} \otimes g \cong I$.

Example: Given a topological space X and $* \in X$, the <u>fundamental 2-group</u> of X has:

- loops based at $* \in X$ as objects, with $\otimes$ being composition of loops



- homotopy classes of homotopies between loops as morphisms:

w/ composition of homotopies as composition:

There's a concept of "equivalence" of monoidal categories (an equivalence of categories that's compatible w/ the monoidal structure, ie tensor product), and thus of 2-groups.

Thrm - Every 2-group is the fundamental 2-group of some topological space, up to equivalence.
(Just like every group is the fundamental group of some space.)

Thrm - Any 2-group is equivalent to one built from these data:

- a group G (this will give the objects of our 2-group w/ mult in G as $\otimes$)
- an abelian group A (this will give the morphisms $a: I \to I$, where I is the unit object in our 2-group, and such morphisms form a group under composition)
- an action ρ of G on A
- a 3-cocycle $c: G \times G \times G \longrightarrow A$ on G valued in the G-module A.
  (This comes from the associator $\alpha_{g,h,k}: (g \otimes h) \otimes k \longrightarrow g \otimes (h \otimes k)$ turned into a morphism $c(g,h,k): I \to I$ in some clever way.)

Proof - Later, but the 3-cocycle equation is equivalent to the pentagon identity for the associator. ▨

This story goes on: you can build an n-group G, A, ρ and an $(n+1)$-cocycle $c: G^{n+1} \longrightarrow A$. What's an n-group? Let's see...

We saw that a <u>strict</u> monoidal category is a 2-category with one object. A general "weak" monoidal category is a weak 2-category with one object, or "bicategory" with one object.
A bicategory is just like a 2-category except composition of morphisms is associative and unital only up to natural isomorphisms:

$\alpha_{f,g,h} : (f\circ g)\circ h \overset{\sim}{\Rightarrow} f\circ(g\circ h)$

$\lambda_f : 1\circ f \overset{\sim}{\Rightarrow} f$

$\rho_f : f\circ 1 \overset{\sim}{\Rightarrow} f$

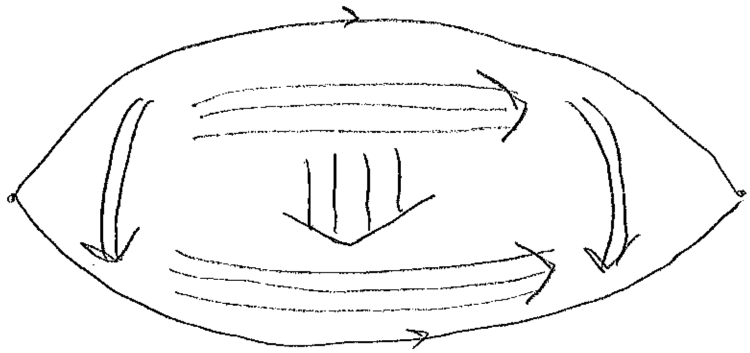
obeying pentagon identity and triangle identity.

In general, weak n-categories are more interesting than strict ones.

THE PERIODIC TABLE OF WEAK n-CATEGORIES

(WEAK) 0-CATEGORIES (sets)	(WEAK) 1-CATEGORIES (categories)	(WEAK) 2-CATEGORIES (bicategories)	(WEAK) 3-CATEGORIES (tricategories)
	with one object	with one object	
monoids	monoidal categories	monoidal bicategories	
Commutative monoids	braided monoidal categories	braided monoidal bicategories	
" "	Symmetric monoidal categories	Sylleptic monoidal bicategories	
" "		Symmetric monoidal bicategories	
" "			
" "			

The Stabilization Hypothesis says that the n-category column stabilizes after n+2 rows. This has been proved by now.



A weak n-category where all j-morphisms ($1 \le j \le n$) are invertible (up to a higher morphism) is called a weak n-groupoid.

Any topological space X has a fundamental n-groupoid $\Pi_n(X)$ with:

- points in X as objects
- paths in X as morphisms
- homotopies between paths as 2-morphisms

  ⋮

- homotopy classes of homotopies of... as n-morphisms.

The Homotopy Hypothesis implies that every n-groupoid is the fundamental n-groupoid of some space, up to equivalence.

Extra hour for Lecture 22 12-7-2018

Why do 2-groups all come, up to equivalence, from quadruples (G, A, ρ, c) where c is a 3-cocycle?

Thrm - Any 2-group is equivalent to one where:

1) The left and right unitors are identity morphisms
2) It's skeletal: isomorphic objects are equal

In this kind of 2-group we can a non-identity associator

$$\alpha_{g,h,k} : (g \otimes h) \otimes k \xrightarrow{\sim} g \otimes (h \otimes k)$$

but by 2 we have

$$(g \otimes h) \otimes k = g \otimes (h \otimes k). \quad ☆$$

Let's consider a 2-group $M = A \times G$ of this form and get (G, A, ρ, c) from it.

Let G be the set of objects, made into a group using $\otimes$. By ☆, $\otimes : G \times G \to G$ really is associative, and def of 2-group says $\forall g \in G$, $\exists g^{-1} \in G$: $g \otimes g^{-1} \cong I \cong g^{-1} \otimes g$, so by 2), $g \otimes g^{-1} = g^{-1} \otimes g$.

Next, let $A = \hom_M(I, I)$

This is a group under composition $\circ : A \times A \to A$ because in a 2-group every morphism is invertible. Why is it an abelian group?

This comes from the Eckmann-Hilton argument.

Note besides

$$\circ : \hom_M(I,I) \times \hom_M(I,I) \longrightarrow \hom_M(I,I)$$

we also have

$$\otimes : \hom_M(I,I) \times \hom_M(I,I) \longrightarrow \hom_M(\underset{I}{\underset{\parallel}{I\otimes I}}, \underset{I}{\underset{\parallel}{I\otimes I}})$$

since $I \otimes I \cong I$

because $\otimes$ is a functor. Functoriality of $\otimes$ also implies

$(a\circ b)\otimes(c\circ d) = (a\otimes c)\circ(b\otimes d)$ "the interchange law"

Also, 1_I is the identity for $\circ$ and $\otimes$ in $A = \hom_M(I,I)$.

Now the Eckmann-Hilton argument kicks in, Given $a, b \in A$,

$$\begin{aligned} a\circ b &= (a\otimes 1_I)\circ(1_I\otimes b) \\ &= (a\circ 1_I)\otimes(1_I\circ b) \\ &= a\otimes b \\ &= (1_I\circ a)\otimes(b\circ 1_I) \\ &= (1\otimes b)\circ(a\otimes 1_I) \\ &= b\circ a \end{aligned}$$

So $\otimes = \circ$ and it's commutative!

Secretly, commutativity is arising from the 2-dimensionality in a bicategory (a 2-group is a bicategory with one object and everything invertible).

So: A is an abelian group.

Next, what's the action $p: G \times A \to A$?

$$p(g)a = 1_g \otimes a \otimes 1_{g^{-1}} : g \otimes I \otimes g^{-1} \to g \otimes I \otimes g^{-1}$$

but $g \otimes I \otimes g^{-1} = I$, so $p(g)a \in A$. You can check

$$p(g)(a \otimes a') = p(g)(a) \otimes p(g)(a')$$
$$p(gg')(a) = p(g)\, p(g')(a)$$

All the objects of $M = A \times G$ are elements of G; what about all the morphisms? They're all isomorphisms and isomorphic objects of M are equal, so they're all automorphisms $f: g \xrightarrow{\sim} g$.

Given any such f, we have

$$f \otimes 1_{g^{-1}} : I \to I$$

so $f \otimes 1_{g^{-1}} =: a \in A$, ie

$$f = a \otimes 1_g \text{ for some } a \in A.$$

So morphisms in $M = A \times G$ correspond to pairs $(a, g) \in A \times G$.

Finally, what about the 3-cocycle $c: G^3 \to A$?

Given $g, h, k \in G^3$, we get

$$\alpha_{g,h,k} : (g \otimes h) \otimes k \to g \otimes (h \otimes k)$$

This corresponds to a pair $(c(g,h,k), (g \otimes h) \otimes k) \in A \times G = M$.

The pentagon identity must be equivalent to some equation involving c.

The pentagon identity says

$$\begin{tikzcd}
 & (g\otimes(h\otimes k))\otimes \ell \arrow[r, "\alpha_{g,h\otimes k,\ell}"] & g\otimes((h\otimes k)\otimes \ell) \arrow[dd, "1_g\otimes \alpha_{h,k,\ell}"] \\
((g\otimes h)\otimes k)\otimes \ell \arrow[ur, "\alpha_{g,h,k}\otimes 1_\ell"] \arrow[dr, "\alpha_{g\otimes h,k,\ell}"'] & & \\
 & (g\otimes h)\otimes(k\otimes \ell) \arrow[r, "\alpha_{g,h,k\otimes \ell}"'] & g\otimes(h\otimes(k\otimes \ell))
\end{tikzcd}$$

In terms of c, this says writing $+$ for group operation in A:

$$c(g,h,k) + c(g,h\otimes k,\ell) + \rho(g)c(h,k,\ell) = c(g\otimes h,k,\ell) + c(g,h,k\otimes \ell)$$

This is the 3-cocycle in group cohomology! Writing group mult. in G in the usual way, not $\otimes$, the above is equivalent to:

$$\rho(g)c(h,k,\ell) - c(gh,k,\ell) + c(g,hk,\ell) - c(g,h,k\ell) + c(g,h,k) = 0 \quad !!$$