Extensions of Groups

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the fundamental Thm. of Arithmetic says any (nonzero) natural number is a product of primes, uniquely up to reordering.

— something similar is true for <u>finite groups</u>, where the "atoms" are the <u>simple</u> groups, those w/o nontrivial normal subgroups.

for finite abelian groups, the simple ones are $\mathbb{Z}_p$ (for p prime); but not every finite abelian group is a product of $\mathbb{Z}_p$. for example, the 4-elm't ab.s are $\mathbb{Z}_2 \times \mathbb{Z}_2$ & $\mathbb{Z}_4$, which is built from two $\mathbb{Z}_2$'s as an "extension": it has $\mathbb{Z}_2$ as a normal subgroup, the quotient of which is $\mathbb{Z}_2$.

— but nonabelian is harder:

Jordan-Hölder Thm: if G is a finite group, there is a <u>composition series</u> for G — subgroups $1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G$ s.t. ① $H_i \triangleleft H_{i+1}$ $(1 \le i < n)$ ② H_{i+1}/H_i is simple.

but, this series is not unique; but the resulting simple groups H_{i+1}/H_i are unique up to reordering.

dfn: a group E is an <u>extension</u> of G by N if $N \triangleleft E$ and $E/N \cong G$; ie

$$0 \to N \hookrightarrow E \twoheadrightarrow G \to 0$$

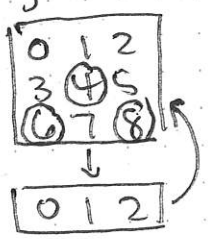
is exact.

So to completely classify finite groups, we "just" need to classify finite simple groups (proof: ~10,000 pages) and then understand extensions. We'll just study <u>abelian</u> extns — $0 \to A \to E \to G \to 0$ where A is abelian. These are classified by $H^2(G, A)$. (*For fully general extns, we'll need "nonabelian cohomology", which uses even more 2-category theory.*)

suppose $1 \to A \xhookrightarrow{i} E \xrightarrow{p} G \to 1$ is an abelian extn; we can think of A as a subgroup of E, so write $i(a) = a$. we can choose a function $j: G \to E$:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, hook, "i"] & E \arrow[r, two heads, "p"] & G \arrow[r] \arrow[l, bend left, "j"] & 1 \end{tikzcd}$$

s.t. $p \circ j = 1_G$. not necessarily a homomorphism, nor unique — but exists: since elmts of G are eA for $e \in E$, we can always choose $j(g)$ to be any elmt of the coset that is g; then $p(j(g)) = g$.



Then each element $e \in E$ is of the form $j(g)\cdot a$ for some unique $g \in G$ & $a \in A$: g says which coset e is in, and a says where e is in the coset. So we get a bijection

$$A \times G \xrightarrow{\sim} E$$
$$(a, g) \mapsto j(g)a$$

(whoops, switch to $a \cdot j(g)$ below)

how does multiplication work in E, in terms of these pairs?

$$j(g)a \cdot j(g')a'$$

$g, g' \in G$
$a, a' \in A$

$$= a\, \underbrace{j(g)\, a'\, j(g)^{-1}}_{\text{A normal } \in A} j(g)\, j(g')$$

call this

$$= a\, \alpha(g)a'\, \underbrace{j(g)\, j(g')\, j(gg')^{-1}}_{\text{this } \in A \text{ too! call it } c(g,g')}\, j(gg')$$

<u>Thm</u>: α is an action of G on A, making A into a G-module. namely

(*HW*)

$$\alpha(g)(aa') = (\alpha(g)a)(\alpha(g)a')$$
$$\alpha(gg')(a) = \alpha(g)(\alpha(g')a)$$

birth of group cohom.

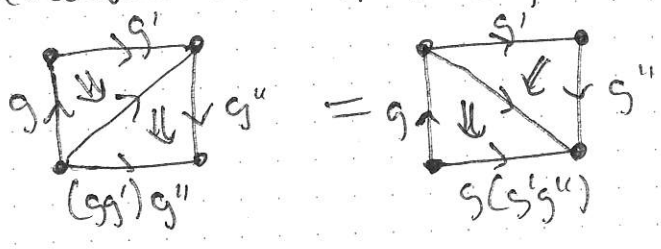
$c: G \times G \to A$ obeys the <u>2-cocycle condition</u>:

$$\alpha(g)c(g', g'') - c(gg', g'') + c(g, g'g'') - c(g, g') = 0$$

(writing G-operation additively, for appearance)

hint: involves associativity...

This is how group cohomology was discovered. The connection to simplices involves the tetrahedron. In G, the associative law:



in $E \cong A \times G$, we have

$$(a,g)(a',g') = (a + \alpha(g)a' + c(g,g'), gg')$$

and associativity of this is connected to the 2-cocycle condition: