The n-Category Café

A group blog on math, physics and philosophy

June 22, 2025

Counting with Categories (Part 1)

Posted by John Baez

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

These are some lecture notes for a $4\frac{1}{2}$-hour minicourse I'm teaching at the **Summer School on Algebra** [https://docs.google.com/spreadsheets/d/1hLTkwT-1F757Cofiyw25BxHPrVXAzg3nqE_XJX_gZNQ/edit?gid=512179038#gid=512179038] at the Zografou campus of the National Technical University of Athens. To save time, I am omitting the pictures I'll draw on the whiteboard, along with various jokes and profoundly insightful remarks. This is just the structure of the talk, with all the notation and calculations.

Long-time readers of the n-Category Café may find little new in this post. I've been meaning to write a sprawling book on combinatorics using categories, but here I'm trying to explain the use of species and illustrate them with a nontrivial example in less than 1.5 hours. That leaves three hours to go deeper.

Part 2 is **here** [https://golem.ph.utexas.edu/category/2025/06/counting_with_categories_part_1.html] , and Part 3 is **here** [https://golem.ph.utexas.edu/category/2025/06/counting_with_categories_part_2.html] .

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Species and their generating functions

Combinatorics, or at least part of it, is the art of counting. For example: how many derangements does a set with n elements have? A **derangement** is a bijection

$$f: S \to S$$

with no **fixed points**, i.e. no points x with $f(x) = x$. There is just one derangement of $S = \{0, 1\}$, namely the function f with

$$f(0) = 1, \qquad f(1) = 0$$

There are 2 derangements of $S = \{0, 1, 2\}$ — can you see what they are? How many derangements are there of a 5-element set? We'll see the answer soon. This is a typical sort of combinatorics question.

But what does counting have to do with category theory? The category of finite sets, FinSet, has

- finite sets as objects

- functions between these as morphisms

What we count, ultimately, are finite sets. Any object $S \in \mathsf{FinSet}$ has a cardinality $|S| \in \mathbb{N}$, so counting a finite set simplifies it to natural number, and the key feature of this process is that

$$S \cong T \Leftrightarrow |S| = |T|$$

We'll count structures on finite sets. A 'species' is roughly a type of structure we can put on finite sets.

Example 1. There is a species of derangements, called D. A D-structure on a finite set S is simply a derangement $f: S \to S$. We write $D(S)$ for the set of all derangements of S. We would like to know $|D(S)|$ for all $S \in \mathsf{FinSet}$.

Example 2. There is a species of permutations, called P. A P-structure on $S \in \mathsf{FinSet}$ is a bijection $f: S \to S$. We know

$$|P(S)| = |S|!$$

We often use n interchangeably for a natural number and a standard finite set with that many elements:

$$n = \{0, 1, \ldots, n-1\}$$

With this notation

$$|P(n)| = n!$$

But what exactly is a species?

Definition. Let E be the category where

- an object is a finite set
- a morphism is a bijection between finite sets

Definition. A **species** is a functor $F: \mathsf{E} \to \mathsf{Set}$. A **tame** species is a functor $F: \mathsf{E} \to \mathsf{FinSet}$.

Any tame species has a **generating function**, which is actually a formal power series $\widehat{F} \in \mathbb{R}[[x]]$, given by

$$\widehat{F} = \sum_{n \geq 0} \frac{|F(n)|}{n!} x^n$$

Example 3. We can compute the generating function of the species of permutations:

$$\widehat{P} = \sum_{n \geq 0} \frac{n!}{n!} x^n = \frac{1}{1-x}$$

Example 4. Given two species F and G there is a species $F \cdot G$ defined as follows. To put an $F \cdot G$-structure on a finite set S is to choose a subset $T \subseteq S$ and put an F-structure on T and a G-structure on $S - T$. We thus have

$$\widehat{F \cdot G} = \sum_{n \geq 0} \frac{|(F \cdot G)(n)|}{n!} x^n$$

$$= \sum_{n \geq 0} \sum_{0 \leq k \leq n} \binom{n}{k} |F(k)||G(n-k)| \frac{x^n}{n!}$$

$$= \sum_{n \geq 0} \sum_{k \geq 0} \frac{|F(k)|}{k!} \frac{|G(n-k)|}{(n-k)!} x^n$$

$$= \sum_{k \geq 0} \sum_{\ell \geq 0} \frac{|F(k)|}{k!} \frac{|G(\ell|}{\ell!} x^k x^\ell$$

$$= \widehat{F}\widehat{G}$$

Example 5. There is a species Exp such that every finite set has a unique Exp-structure! We thus have

$$|\mathrm{Exp}(S)| = 1$$

for all $S \in \mathsf{FinSet}$, so

$$\widehat{\mathrm{Exp}} = \sum_{n \geq 0} \frac{1}{n!} x^n = \exp x$$

That's why we call this boring structure an Exp-structure. I like to call Exp **being a finite set**. Every finite set has the structure of being a finite set in exactly one way.

Example 6. Recall that a D-structure on a finite set is a derangement of that finite set. To choose a permutation $f \colon S \to S$ of a finite set S is the same as to choose a subset $T \subset S$, which will be the set of fixed points of f, and to choose a derangement of $S - T$. Thus by Example 4 we have

$$P \cong \mathrm{Exp} \cdot D$$

and also

$$|P| = |\mathrm{Exp}| \cdot |D|$$

so by Example 3 and Example 5 we have

$$\frac{1}{1-x} = \exp(x)|D|$$

so

$$|D| = \frac{e^{-x}}{1-x}$$

or

$$\sum_{n \geq 0} \frac{|D(n)|}{n!} x^n = \frac{e^{-x}}{1-x}$$

$$= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)(1 + x + x^2 + x^3 + x^4 + \cdots)$$

From this it's easy to work out $|D(n)|$. I'll do the example of $n = 5$. If you think about the coefficient of x^5 in the above product, you'll see it's

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}$$

So we must have

$$\frac{|D(5)|}{5!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}$$

or

$$\begin{aligned} |D(5)| &= 5!\left(\tfrac{1}{2!} - \tfrac{1}{3!} + \tfrac{1}{4!} - \tfrac{1}{5!}\right) \\ &= 3 \cdot 4 \cdot 5 - 4 \cdot 5 + 5 - 1 \\ &= 60 - 20 + 5 - 1 \\ &= 44 \end{aligned}$$

In general we see

$$|D(n)| = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{n!}\right)$$

But we can go further with this… and I'll ask you to go further in this homework:

Exercise 1. Do the problems here:

- **Let's get deranged!** [https://math.ucr.edu/home/baez/qg-winter2004/derangement.pdf]

The category of species

You'll notice that above I said there was an isomorphism

$$P \cong \mathrm{Exp} \cdot D$$

without ever defining an isomorphism of species! So let's do that. In fact there's a category of species.

I said a species is a functor

$$F \colon \mathsf{E} \to \mathsf{Set}$$

where E is the category of finite sets and bijections. But what's a morphism between species?

It's easy to guess if you know what goes between functors: it's a natural transformation.

Definition. For any categories C and D, let the **functor category** D^{C} be the category where

- an object is a functor $F\colon \mathsf{C} \to \mathsf{D}$
- morphism $\alpha\colon F \Rightarrow G$ is a natural transformation.

(We write this with double arrows since a functor was already a kind of arrow, and now we're talking about an arrow between arrows.)

Definition. The **category of species** is $\mathsf{Set}^{\mathsf{E}}$. The **category of tame species** is $\mathsf{FinSet}^{\mathsf{E}}$.

Two tame species can have the same generating function but not be isomorphic! You can check this in these exercises:

Exercise 2. In Example 2 we began defining the species of permutations, P. We said that for any object $S \in \mathsf{E}$, $P(S)$ is the set of permutations of S. But to make P into a functor we also need to say what it does on morphisms of math E. That is, given a bijection $f\colon S \to S'$ and a permutation $g \in F(S)$ we need a way to get a permutation $F(f)(g)\colon S' \to S'$. Figure out the details and then show that $P\colon \mathsf{E} \to \mathsf{Set}$ obeys the definition of a functor:

$$P(f \circ g) = P(f) \circ P(g)$$

for any composable pair of bijections $g\colon S \to S'$, $f\colon S' \to S''$, and

$$P(1_S) = 1_{P(S)}$$

Exercise 3. Show that there is a species L, the **species of linear orderings**, such that an L-structure on $S \in \mathsf{FinSet}$ is a linear ordering on S. In other words: first let $L(S)$ be the set of linear orderings. Then for any bijection $f\colon S \to S'$ define a map $L(f)\colon L(S) \to L(S')$ that sends linear orderings of S into linear orderings of S'. Then show that $L\colon \mathsf{E} \to \mathsf{Set}$ obeys the definition of a functor.

Exercise 4. Now show there is no natural isomorphism $\alpha\colon P \to L$. However we have

$$|P(S)| = |L(S)| = |S|!$$

for any finite set S, so P and L have the same generating function:

$$\hat{P} = \hat{L}$$

Thus, you've found nonisomorphic tame species with the same generating function!

This may make you sad, because you might hope that you could tell whether two species were isomorphic just by looking at their generating function. But if that were true, species would contain no more information than formal power series. In fact they contain more! Species are like an improved version of formal power series. And there's not just a set of them, there's a *category* of them.

Posted at June 22, 2025 2:59 PM UTC

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Re: Counting with Categories (Part 1)

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

I remember doing this homework assignment two decades ago! I hope that you write that book.

Typo: In discussing the product of two species, at one point you mention X, but there is no X there; you mean T.

Formatting (itex) issue: Vertical bars are rendered as operators when they should usually be delimiters (or at least ordinary characters), creating extra space in most contexts. A quick way to fix this is to always put curly braces around them: `{|X|} = {|Y|} = {|Z|}` produces $|X| = |Y| = |Z|$ (whereas `|X| = |Y| = |Z|` produces $|X \mid = \mid Y \mid = \mid Z|$ with extra space except at the endpoints).

And you probably anticipated this as soon as you saw that it was me making a comment on this post, but the formula for $|D(n)|$ is missing the first two terms, $\frac{1}{0!}$ and $-\frac{1}{1!}$. Sure, they cancel, but not for $n = 0$, where $\frac{1}{0!}$ is both the first and last term and $-\frac{1}{1!}$ doesn't appear. Then you need $\frac{1}{0!}$ to get the correct answer, $|D(0)| = 1$. (The unique function from the empty set to itself is a bijection with no fixed points.)

Posted by: Toby Bartels on June 27, 2025 9:15 AM | Permalink | Reply to this

Re: Counting with Categories (Part 1)

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Thanks! I've been annoyed by the spacing around | for years, but never enough to figure out how to fix it. I'm not sure I'll always have the energy to do it, but I tried it out and yes, it works.

Posted by: John Baez on June 27, 2025 9:44 AM | Permalink | Reply to this

Re: Counting with Categories (Part 1)

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

My flight to Athens was delayed and I wound up taking a taxi to my hotel at midnight. Out of the blue, after no conversation, the driver asked me if I was a mathematician or philosopher. When I admitted to the former, he tapped his temple and said "they've got it going on there, no?" His smile was impish yet benign.

When I arrived, I was so exhausted I couldn't figure out how to turn on the shower and simply went to sleep. The next day I sheepishly asked the guy at the front desk to show me how.

Later that day my key card didn't work, so I went to the same guy and said "I think my key card has become demagnetized."

He said "Are you an engineer?"

If this was an insult it was delivered in the most dead-pan manner possible, so I decided to take it as a serious question and said "No, a mathematician". He replied "So was my father", and something let me know he was not making fun of me.

Based on this insignificant sample I conclude that the Greeks, or at least the taxi drivers and hotel desk clerks among them, enjoy guessing people's occupations from their looks, and are not bad at it. Also, they are less anti-intellectual than Americans: they do not say "oh, I always hated mathematics" when they hear what I do.

Posted by: John Baez on June 30, 2025 2:50 PM | Permalink | Reply to this

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The n-Category Café

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June 26, 2025

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Posted by John Baez

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

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The 2-rig of species

We've seen that the category of tame species $\mathsf{FinSet}^{\mathsf{E}}$ closely resembles the ring of formal power series $\mathbb{R}[[x]]$, but it's richer because two nonisomorphic tame species can have the same generating function. To dig deeper we need to understand this analogy better. But what do I mean exactly?

For starters, we can add and multiply species:

Addition. Given species $F, G \colon \mathsf{E} \to \mathsf{Set}$, they have a **sum** $F + G$ with

$$(F+G)(S) = F(S) + G(S)$$

where at right the plus sign means the disjoint union, or **coproduct** [https://en.wikipedia.org/wiki/Coproduct] , of sets. Here I've said what $F + G$ does to objects $S \in \mathsf{FinSet}$, but it does something analogous to morphisms — figure it out and check that it makes $F + G$ into a functor! If F and G are tame, so is $F + G$, and

$$\widehat{F+G} = \hat{F} + \hat{G}$$

Multiplication. Given species $F, G \colon \mathsf{E} \to \mathsf{Set}$, I said last time that we can multiply them and get a species $F \cdot G$ with

$$(F \cdot G)(S) = \sum_{X \subseteq S} F(X) \times G(S - X)$$

Again, I'll leave it to you to guess what $F \cdot G$ does to morphisms and check that $F \cdot G$ is then a functor. If F and G are tame, so is $F \cdot G$, and

$$\widehat{F \cdot G} = \hat{F} \cdot \hat{G}$$

Exercise 5. If you know about **coproducts** [https://en.wikipedia.org/wiki/Coproduct] , you can show that $F + G$ is the coproduct of F and G in the category of species, $\mathsf{Set}^{\mathsf{E}}$, which we defined last time.

Exercise 6. If you know about **products** [https://en.wikipedia.org/wiki/Product_(category_theory)] , you can show that $F \cdot G$ is *not* the product of F and G in the category of species. Thus, we often call it the **Cauchy product.** But if you know about **symmetric monoidal categories** [https://en.wikipedia.org/wiki/Symmetric_monoidal_category] , you can with some significant work show that the Cauchy product makes $\mathsf{Set}^{\mathsf{E}}$ into a symmetric monoidal category. For starters, this means that it's associative and commutative up to isomorphism—but it means more than just that. (If you know about **Day convolution** [https://en.wikipedia.org/wiki/Day_convolution] , this can speed up your work.)

Now let's think about addition and the Cauchy product together. We can check that

$$F \cdot (G + H) \cong F \cdot G + F \cdot H$$

and indeed addition and the Cauchy product give the category of species a structure much like that of a ring! It's even more like a 'rig', which is a 'rings without negatives', since we can add and multiply species, but not subtract them. But all the rig laws hold, not as equations, but as natural isomorphism.

In fact the binary coproduct is just a special case of something called a **colimit** [https://en.wikipedia.org/wiki/Limit_(category_theory)#Colimits] , defined by a more general universal property. And the Cauchy product distributes over all colimits! We thus say the category of species is a 'symmetric 2-rig'.

A bit more precisely:

Definition. A **2-rig** is a monoidal category $(\mathsf{R}, \otimes, I)$ with all colimits, such that the tensor product $\otimes$ distributes over colimits in each argument. That is, if D is any small category and $F \colon \mathsf{D} \to \mathsf{R}$ is any functor, for any object $r \in \mathsf{R}$, the natural morphisms

$$\begin{array}{ccc} \mathrm{colim}_{i \in \mathsf{D}}(r \otimes F(i)) & \longrightarrow & r \otimes (\mathrm{colim}_{i \in \mathsf{D}} F(i)) \\ \mathrm{colim}_{i \in \mathsf{D}}(F(i) \otimes r) & \longrightarrow & (\mathrm{colim}_{i \in \mathsf{D}} F(i)) \otimes r \end{array}$$

are isomorphisms.

Definition. A **symmetric 2-rig** is a 2-rig whose underlying monoidal category is a symmetric monoidal category.

One can work through the details of these definitions and show the category of species is a symmetric 2-rig. But something vastly better is true. It's very similar to the ring of polynomials in one variable!

Why is the ring $\mathbb{Z}[x]$, the ring of polynomials in one variable with integer coefficients, so important in mathematics? Because it's the free ring on one generator! That is, given any ring R and any element $r \in R$, there's a unique ring homomorphism

$$f \colon \mathbb{Z}[x] \to R$$

with

$$f(x) = r$$

To see this, note that the homomorphism rules force that for any $P \in \mathbb{Z}[x]$ we have

$$f(P) = P(r)$$

By the way, $\mathbb{Z}[x]$ is also the free *commutative* ring on one generator, and the 2-rig of species is similar: it's the free symmetric 2-rig on one object X. But what is this object X?

X is a particular species—a structure you can put on finite sets. It's the structure of 'being a 1-element set'. What I mean is this: it's the structure that you can put on a finite set S in a unique way if S has one element, and not at all if S has any other number of elements. In other words:

$$X(S) = \begin{cases} 1 & \text{if} \quad |S| = 1 \\ \varnothing & \text{if} \quad |S| \neq 1 \end{cases}$$

The name X is appropriate because the generating function of this species is x:

$$\widehat{X} = \sum_{n \geq 0} \frac{|X(n)|}{n!} x^n = x$$

Here's the big theorem, which I won't prove here:

Theorem. The symmetric 2-rig of species, $\mathsf{Set}^{\mathsf{E}}$, is the free symmetric 2-rig on one generator. That is, for any symmetric 2-rig R and any object $r \in \mathsf{R}$, there exists a map of 2-rigs

$$f \colon \mathsf{Set}^{\mathsf{E}} \to \mathsf{R}$$

such that $f(X) = r$, and f is unique up to natural isomorphism.

This result has a baby brother, too. First, note that the free ring on *no* generators is $\mathbb{Z}$. That is, $\mathbb{Z}$ is the **initital** [https://en.wikipedia.org/wiki/Initial_and_terminal_objects] ring: for any ring R there exists a unique ring homomorphism

$$f \colon \mathbb{Z} \to R$$

$\mathbb{Z}$ is also the free commutative ring on no generators. This should make us curious about the free symmetric 2-rig on no generators. This is the category of sets, with its cartesian product as the symmetric monoidal structure.

Theorem. The symmetric 2-rig of sets, Set, is the initial 2-rig. That is, for any symmetric 2-rig R there exists a map of 2-rigsar

$$f \colon \mathsf{Set} \to \mathsf{R}$$

which is unique up to natural isomorphism.

So we have a wonderful analogy:

> **The category of sets is the free symmetric 2-rig on no generator, just as $\mathbb{Z}$ as the free commutative ring on no generators.**

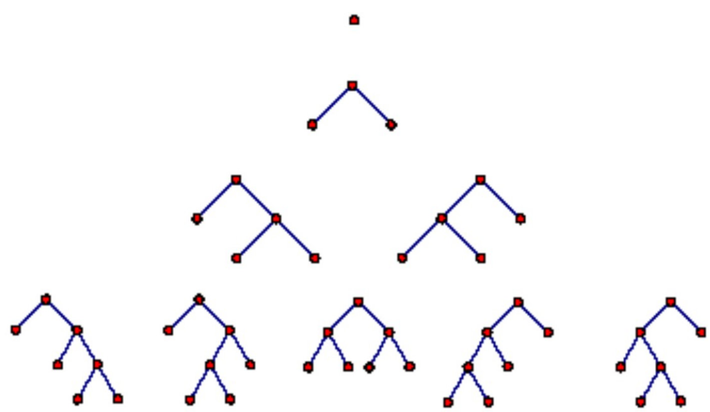
> **The category of species is the free symmetric 2-rig on one generator, just as $\mathbb{Z}[x]$ is the free commutative ring on one generator.**

This suggests that there's a lot more one can do to 'categorify' ring theory — that is, to take ideas from ring theory and develop analogous ideas in 2-rig theory. And even better, it shows that *a lot of combinatorics is categorified ring theory!*

Counting binary trees

But let's see how we can use this 2-rig business to count things. Let's make up a species B where a B-structure on a finite set S is a making it into the leaves of a rooted planar binary tree. More precisely, it's a bijection between S and the set of leaves of some rooted planar binary tree.

Here are some rooted planar binary trees:



There's 1 rooted planar binary tree with 1 leaf, 1 with 2, 2 with 3, 5 with 4... and so on.

I'll draw lots of pictures to explain these trees and their leaves, but the quickest definition of B is recursive, involving no pictures.

To put a B structure on a finite set S, we either

- put an X-structure on it

or

- partition it into two parts T and $S - T$ and put a B-structure on each part.

Remember, an X-structure on S is the structure of 'being a one-element set'. This handles the case of the binary tree with just one leaf. All other binary trees consist of two binary trees glued together at their root.

We can state this recursive definition as an equation:

$$B(S) = X(S) + \sum_{T \subseteq S} B(T) \times B(T - S)$$

and we can express this more efficiently using the sum and Cauchy product of species:

$$B = X + B \cdot B$$

We can now use this to count B-structures on a finite set! By taking generating functions we get

$$\hat{B} = x + \hat{B}^2$$

or

$$\hat{B}^2 - \hat{B} + x = 0$$

This is a quadratic equation for the formal power series $\hat{B}$, which we can solve using the quadratic formula!

$$\hat{B} = \frac{1 \pm \sqrt{1 - 4x}}{2}$$

Since the coefficients of a generating function can't be negative, we must have

$$\hat{B} = \frac{1 - \sqrt{1 - 4x}}{2}$$

since the other sign choice gives a term proportional to x with a negative coefficient. Using a computer we can see

$$\frac{1 - \sqrt{1 - 4x}}{2} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + 429x^8 + \cdots$$

Remember

$$\hat{B} = \sum_{n \geq 0} \frac{|B(n)|}{n!} x^n$$

so we get

$$\frac{|B(0)|}{0!} = 0$$

$$\frac{|B(1)|}{1!} = 1$$

$$\frac{|B(2)|}{2!} = 1$$

$$\frac{|B(3)|}{3!} = 2$$

$$\frac{|B(4)|}{4!} = 5$$

$$\frac{|B(5)|}{5!} = 14$$

and so on. The factorials arise because there are $n!$ ways to label the leaves of a planar binary tree with n leaves by elements of $\{0, \ldots, n-1\}$. The interesting part is the sequence

$$0, 1, 1, 2, 5, 14, \ldots$$

These give the number of planar binary trees with n leaves! They're called the **Catalan numbers** [https://en.wikipedia.org/wiki/Catalan_number] .

Exercise 7. Use your mastery of Taylor series to show that

$$\frac{1 - \sqrt{1-4x}}{2} = \sum_{n \geq 0} \frac{2^{n-1}(2n-3)!!}{n!} x^n$$

so

$$\frac{|B(n)|}{n!} = \frac{2^{n-1}(2n-3)!!}{n!}$$

Let's check this for $n = 4$:

$$\frac{2^{4-1}(2 \cdot 4 - 3)!!}{4!} = \frac{8 \cdot 5!!}{4!} = \frac{8 \cdot 5 \cdot 3 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} = 5$$

which matches how there are 5 rooted planar binary trees with 4 leaves!

Posted at June 26, 2025 1:00 PM UTC

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Posted by John Baez

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Here's my third and final set of lecture notes for a $4\frac{1}{2}$-hour minicourse at the **Summer School on Algebra** [https://docs.google.com/spreadsheets/d/1hLTkwT-1F757Cofiyw25BxHPrVXAzg3nqE_XJX_gZNQ/edit?gid=512179038#gid=512179038] at the Zografou campus of the National Technical University of Athens.

Part 1 is **here** [https://golem.ph.utexas.edu/category/2025/06/counting_with_categories_part.html] , and Part 2 is **here** [https://golem.ph.utexas.edu/category/2025/06/counting_with_categories_part_1.html] .

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Last time I began explaining how a chunk of combinatorics is categorified ring theory. Every structure you can put on finite sets is a species, and the category of species is the free symmetric 2-rig on one object, just as the polynomial ring $\mathbb{Z}[x]$ is the free commutative ring on one generator.

In fact it's almost true that every species gives an element of $\mathbb{Z}[x]$! You should think of mathematics as wanting this to be true. But it's not quite true: in fact every *tame* species has a generating function, which is an element of $\mathbb{R}[[x]]$, a ring that's a kind of 'completion' of $\mathbb{Z}[x]$. There's a lot to be said about the slippage here, and why it's happening, but there's not time for such abstract issues now. Instead, let's see what we can do with this analogy:

> Just as $\mathbb{Z}[x]$ is the free commutative ring on one generator, the category of species $\mathsf{Set}^{\mathsf{E}}$ is the free symmetric 2-rig on one generator.

Substitution of species

$\mathbb{Z}[x]$ is the free commutative ring on one generator, namely x. Thus for any element $P \in \mathbb{Z}[x]$, there's a unique ring homomorphism

$$f\colon \mathbb{Z}[x] \to \mathbb{Z}[x]$$

sending x to P:

$$f(x) = P$$

What is this homomorphism f? It sends x to P, so it must send x^2 to P^2, and it must send $x^2 + 3x + 1$ to

$P^2 + 3P + 1$, and so on. Indeed it sends any polynomial Q to $Q(P)$, or in other words $Q \circ P$:

$$f(Q) = Q \circ P$$

So, we're seeing that we can *compose* elements of $\mathbb{Z}[x]$, or substitute one in another.

The same thing works for species! Since the category of species, $\mathsf{Set}^{\mathsf{E}}$, is the free symmetric 2-rig on one generator X, for any species $P \in \mathsf{Set}^{\mathsf{E}}$ there's a unique map of symmetric 2-rigs

$$F \colon \mathsf{Set}^{\mathsf{E}} \to \mathsf{Set}^{\mathsf{E}}$$

sending X to P:

$$F(X) = P$$

And following the pattern we saw for polynomials, we can say

$$F(Q) = Q \circ P$$

But this time we are *defining* the $\circ$ operation by this formula, since we didn't already know a way to compose species, or substitute one in another. But it's very nice:

Theorem. If Q and P are species, to put a $Q \circ P$-structure on a finite set S is to choose an unordered partition of S into nonempty sets $T_1, \dots, T_n$:

$$S = T_1 \cup \cdots T_n, \qquad i \neq j \Rightarrow T_i \cap T_j = \emptyset$$

and put a P-structure on $\{1, \dots, n\}$ and a Q-structure on each set T_i. Moreover

$$\widehat{Q \circ P} = \widehat{Q} \circ \widehat{P}$$

if the constant term of $\widehat{P} \in \mathbb{R}[[x]]$ vanishes.

Let's illustrate this with an easy example and then try some a more interesting example.

Example 7. Remember from **Example 5** [https://golem.ph.utexas.edu/category/2025/06/counting_with_categories_part.html] that Exp is our name for the species is **being a finite set**—every finite set has this structure in exactly one way. We call it Exp because its generating function is the power series for the exponential function:

$$\widehat{\mathrm{Exp}} = \sum_{n \geq 0} \frac{x^n}{n!}$$

Let $\frac{X^2}{2!}$ be the species **being a 2-element set**: every set with 2 elements has this structure in exactly one way, while a set with any other number of elements cannot have this structure. We give it this funny name because

$$\widehat{\frac{X^2}{2!}} = \sum_{n \geq 0} \frac{a_n}{n!} x^n$$

where

$$a_n = \begin{cases} 1 & \text{if} \quad n = 2 \\ 0 & \text{if} \quad n \neq 2 \end{cases}$$

so

$$\widehat{\frac{X^2}{2!}} = \frac{x^2}{2}$$

Now let's compose these species, and let

$$F = \text{Exp} \circ \frac{X^2}{2!}$$

By the theorem, to put an F-structure on S is to partition S into nonempty parts, put an Exp-structure on the set of parts, and put an $\frac{X^2}{2!}$-structure on each part. But there's just one way to put an Exp-structure on the set of parts. So, putting an F-structure on S is the same as partitioning S into parts of cardinality 2.

How many ways are there to do that? We could figure it out, but let's use the theorem, which says

$$\hat{F} = \widehat{\text{Exp}} \circ \widehat{\frac{X^2}{2!}}$$

and thus

$$\hat{F} = e^{x^2/2}$$

so

$$\begin{aligned} \sum_{n \geq 0} \frac{|F(n)|}{n!} &= e^{x^2/2} \\ &= 1 + (x^2 / 2) + \frac{(x^2 / 2)^2}{2!} + \frac{(x^2 / 2)^3}{3!} + \cdots \\ &= \sum_{k \geq 0} \frac{1}{2^k k!} x^{2k} \end{aligned}$$

So, we see that if $n = 2k$ is even,

$$|F(n)| = \frac{n!}{2^k k!}$$

So this is the number of F-structures on an n-element set. We can check our work if we know a bit about symmetries and counting. The group S_n acts transitively on the set $F(n)$ of ways to partition an n-element set into 2-element parts. The subgroup that fixes any such partition is $S_k \ltimes (\mathbb{Z} / 2)^k$, since we can permute the k parts and also permute the 2 elements in each part. Thus,

$$|F(n)| = \frac{n!}{2^k k!}$$

But the generating function method is nice because we can just turn the crank. And as we'll see, we can use it to solve harder problems.

Example 8. How many ways can we partition an n-element set into nonempty parts? This is called the nth **Bell number** [https://en.wikipedia.org/wiki/Bell_number] , b_n. Let's see if we can calculate it!

Let Part be the species of partitions. I claim there are species Exp and NE such that

$$\mathrm{Part} \cong \mathrm{Exp} \circ \mathrm{NE}$$

Exp is our old friend 'being a finite set'. NE is the species **being a nonempty finite set**. To partition a finite set into nonempty finite sets is to partition it into sets, put a Exp-structure on the set of parts, and put an NE-structure on each part. That gives the isomorphism above!

We know that

$$\widehat{\mathrm{Exp}} = \sum_{n \geq 0} \frac{x^n}{n!} = \exp(x)$$

and similarly

$$\widehat{\mathrm{NE}} = \sum_{n \geq 1} \frac{x^n}{n!} = \exp(x) - 1$$

Thus by the theorem,

$$\widehat{\mathrm{Part}} = \widehat{\mathrm{Exp}} \circ \widehat{\mathrm{NE}} = e^{e^x - 1}$$

So by the definition of the Bell numbers,

$$\sum_{n \geq 0} \frac{b_n}{n!} x^n = e^{e^x - 1}$$

This is not quite an explicit formula for the Bell numbers, but it's almost as good! Let's use it to calculate the first few Bell numbers. We'll work out the power series for $e^{e^x - 1}$ only up to order x^3:

$$\begin{aligned} e^{e^x - 1} &= \exp\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) \\ &= 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) + \frac{1}{2}\left(x + \frac{x^2}{2} + \cdots\right)^2 + \frac{1}{6}x^3 + \cdots \\ &= 1 + x + \frac{2}{2!}x^2 + \frac{5}{3!}x^3 + \cdots \end{aligned}$$

This gives

$$b_0 = 1, \quad b_1 = 1, \quad b_2 = 2, \quad b_3 = 5$$

which we can easily confirm. So it works!

There is a *lot* more one can do with species. Luckily there are some free books that will take you further:

- François Bergeron, Gilbert Labelle, and Pierre Leroux, ***Introduction to the Theory of Species of Structures*** [http://bergeron.math.uqam.ca/wp-content/uploads/2013/11/book.pdf] .
- Phillipe Flajolet and Robert Sedgewick, ***Analytic Combinatorics*** [http://algo.inria.fr/flajolet/Publications/book.pdf] .
- Herbert S. Wilf, ***Generatingfunctionology*** [https://www.math.upenn.edu/~wilf/gfology2.pdf] .

Linear species

Finally, let me just conclude by mentioning a variant. We can define the category of **linear species** to be $\mathsf{Vect}^{\mathsf{E}}$ where Vect is the category of complex vector spaces and linear maps. So, it's just like the category of species but with Set replaced by Vect.

It turns out that like the category of species, the category of linear species is a 2-rig. We call the addition $\oplus$ because now, given two linear species $F, G \colon \mathsf{E} \to \mathsf{Vect}$, we define

$$(F \oplus G)(S) = F(S) \oplus G(S)$$

).

But it's still just the coproduct. The Cauchy product of linear species is defined by

$$(F \cdot G)(S) = \sum_{X \subseteq S} F(X) \otimes G(S - X)$$

We can also compose linear species! Everything is very similar. But now it's connected to the representation theory of the symmetric group, because the category E is equivalent to the coproduct of all the 1-object categories BS_n corresponding to the symmetric groups, and a functor from BS_n to Vect is the same as a representation of S_n. And because representations of the symmetric group are classified by Young diagrams, so are linear species.

I won't go into this more now, but my friends Joe and Todd and I have studied this subject in great detail, and it connects many areas of mathematics in a delightful way:

- John Baez, Joe Moeller and Todd Trimble, **Schur functors and categorified plethysm** [https://arxiv.org/abs/2106.00190] .
- John Baez, Joe Moeller and Todd Trimble, **2-Rig extensions and the splitting principle** [https://arxiv.org/abs/2410.05598] .

For example, the second paper here relates linear species to the ring of **symmetric functions** [https://en.wikipedia.org/wiki/Symmetric_function] , another popular and fundamental topic in combinatorics.

Posted at June 26, 2025 1:03 PM UTC

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Tracked: June 27, 2025 9:12 AM

Re: Counting with Categories (Part 3)

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

james dolan should complain that you misspelled generatingfunctionology as Generatingfunctionology.

Say a species S has a generating function g, and g satisfies a linear differential equation p with polynomial coefficients, i.e. p(g)=0. (I think about D-modules from time to time.) For example, e^x is annihilated by p = 1 y' - 1 y. Is there a nice categorification of "p(g)=0" that one can state directly about S?

Posted by: Allen Knutson on July 2, 2025 5:30 AM | Permalink | Reply to this

Re: Counting with Categories (Part 3)

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

When I first met James online on sci.math, he always typed in all caps. I eventually told him that made him look like a crank. He told me he'd think about that and maybe do something about it. A few days later he started writing in all lower case, and he has ever since. I think he mainly dislikes unnecessary fiddling with typography, finding it distracting.

Publications seem to differ [https://www.tandfonline.com/doi/pdf/10.1080/00029890.1990.11995676] on whether the book's title really is *generatingfunctionology*, or whether that just happens to be the font of the title on the cover of the book (a subtle distinction).

The best way to write the differential equation obeyed by a species S is not

$$P(S) = 0$$

but

$$S \cong Q(S)$$

where Q is some polynomial coefficient differential operator with $Q \in \mathbb{N}[X, \frac{d}{dX}]$. This gives a recursive definition of the species S, very much like the recursive definition I gave in Part 2 of the species of rooted labeled binary trees:

$$B \cong X + B^2$$

Species obeying such recursive definitions tend to describe tree-like structures, and they're studied extensively in this book:

- François Bergeron, Gilbert Labelle and Pierre Leroux. *Combinatorial Species and Tree-like Structures*, Cambridge University Press, 1998.

though honestly I forget how far they go in relating properties of species to properties of the differential equations they obey. I am very interested in understanding the branch cut in the solutions to

$$B \cong X + B^2$$

and how the theory of branched coverings of $\mathbb{C}$ is related to species, but I haven't gotten very far yet. Maybe I should try to categorify the theory of D-modules, instead.

Posted by: John Baez on July 2, 2025 5:59 AM | Permalink | Reply to this

Re: Counting with Categories (Part 3)

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Where does the tree-like structure story relate to Lagrange inversion, as in **More secrets of the associahedra** [https://golem.ph.utexas.edu/category/2018/01/more_secrets_of_the_associahed.html] ? Maybe I'm asking about categorifying that formula.

Also, is the Lagrange inversion formula about the formal group of $\mathrm{Diff}(S^1)$, and if so, does the tree-based (or associahedron-based) formula for the inverse have an extension to Virasoro? At which point I'd also want to know how to involve the formal Virasoro group in the Lazard ring story.

Posted by: Allen Knutson on July 2, 2025 3:23 PM | Permalink | Reply to this

Re: Counting with Categories (Part 3)

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Good questions! I don't really understand that formula for the compositional inverse of a formal power series in terms of associahedra… so before I could give any interesting answer, I'd have to think about that. I'll just note that if we want to *categorify* this formula, we'll need to handle formal power series with negative coefficients. The generating function for a species has only nonnegative coefficients. One way to get a formal power series with negative coefficients is from the generating function of a 'derived species' — e.g. a functor F from the groupoid of finite sets not to FinSet but to the category of bounded chain complexes of finite-dimensional vector spaces, with quasi-isomorphisms inverted. The idea is to use the Euler characteristic as a substitute for cardinality in defining the generating function:

$$\hat{F} = \sum_{n \geq 0} \frac{\chi(F(n))}{n!} x^n$$

where $F(n)$ is the chain complex that F associates to the n-element set.

(Lately I've been working on derived species with Todd Trimble, so they're on my mind.)

Posted by: John Baez on July 2, 2025 3:31 PM | Permalink | Reply to this

Re: Counting with Categories (Part 3)

[http://golem.ph.utexas.edu/~distler/blog/mathml.html]

Looking a bit more at the paper we're talking about

- Marcelo Aguiar, Federico Ardila, **Hopf monoids and generalized permutahedra** [https://arxiv.org/abs/1709.07504] .

it's clear that my Euler characteristic idea was on the right track, but the generating function idea was not.

I don't understand most of what's going on, but it's fun to seek the key point at which counting faces in an associahedron becomes important for inverting formal power series. (Here I mean either inverting with respect to ordinary multiplication, *or* with respect to composition: the associahedron shows up for *both!*)

I think it's Theorem 7.1, which gives a formula involving a sum over faces q of the associahedron, weighted by $(-1)^{\dim(\mathsf{q})}$. That's already very reminiscent of an Euler characteristic, but in the proof they come out and say

> We would like to interpret this as an Euler characteristic, but….

and then point out an obstacle, and then overcome this obstacle.

But they are not, as I'd guessed above, interpreting this sum in terms of the generating function of some derived species! Instead they are interpreting it as the antipode in a certain specific Hopf monoid in the category of "vector species", i.e. the category of Vect-valued species, i.e. the category $\mathsf{Vect}^{\mathsf{E}}$ where E is the groupoid of finite sets.

Since the antipode is a *morphism* in the category of vector species, rather than an *object*, this makes it much easier for it to involve minus signs. However, I don't know why this morphism should be related to the associahedron, since I don't yet understand the proof of Theorem 7.1.

Theorem 7.1 is proved using a simpler formula for the antipode, which they call "Takeuchi's formula"—see Definition 2.11. This is a general formula giving any connected bimonoid H in vector species an antipode, and thus making it a Hopf monoid.

("Connected" here means that the vector space $H(\emptyset)$ is the ground field. This should remind you of **the definition of a connected $\mathbb{N}$-graded algebra** [https://ncatlab.org/nlab/show/graded+algebra#definition] .)

Posted by: John Baez on July 2, 2025 6:39 PM | Permalink | Reply to this

Read the post **Counting with Categories (Part 2)** [https://golem.ph.utexas.edu/category/2025/06/counting_with_categories_part_1.html]
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