<u>Combinatorics</u> : the art of counting. 9/26

the study of structures on finite sets.

$F :=$ [finite sets, functions].

any finite set X has a cardinality $|X| \in \mathbb{N}$, its number of components.

why do we count?

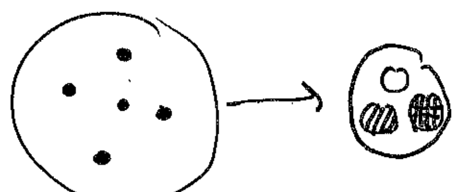
$$\left(\begin{matrix} X \cong Y \\ \exists \text{ iso } f: X \to Y \\ \text{in } F \end{matrix}\right) \overset{\text{decategorification}}{\Longleftrightarrow} \left(\begin{matrix} |X| = |Y| \\ \text{equality in } \mathbb{N} \end{matrix}\right)$$

<u>sneaky</u>: forks + families

also useful to go the <u>other</u> way.

what is "structure"?

ex 1) if $k \in \mathbb{N}$, a <u>k-coloring</u> of a finite set X with $c: X \longrightarrow \{1, \ldots, k\}$



how many k-colorings of X? $k^{|X|}$.

(define $Y^X := \{f : X \to Y\}$; then $|Y^X| = |Y|^{|X|}$.)

similarly $|X \times Y| = |X| \times |Y|$

Baez: grew up during "New Math", when they actually explained stuff.

$|X+Y| = |X|+|Y|$, "initial" $|\{\ \}| = 0$ $(\emptyset \in F)$

"coproduct" "terminal" $|\{\emptyset\}| = 1$. $(\forall x = x)$

2) a <u>pointed</u> finite set is a finite set X equipped with a function $p: \{x\} \to X$.

how many points? $|X|$.

3) a <u>partition</u> of a finite set X

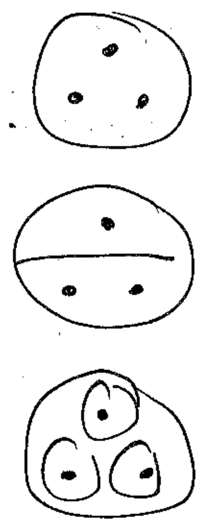
(Gabe: same as coloring? not quite - needs to be <u>surjective</u>)

is a collection of disjoint nonempty subsets such that $X = \bigcup X_i$.

Q: how many partitions does X have?

if $|X| = n$, then it's the nth Bell number.

n	b_n	
0	1	1! collection of parts is empty.
1	1	
2	2	
3	5	



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What is a kind of structure on finite sets?

* André Joyal: "species"

Let $\mathcal{S} = [\text{finite sets, bijections}]$.

<u>def</u> a (finite) species is a functor

$$G : \mathcal{S} \to \mathcal{F}.$$

$G(X) \in \mathcal{F}$ is the set of all structures of kind "G" that you can put on X.

(e.g. 3-colorings, $G(X) = \{1,2,3\}^X$)

given $f : X \to Y$ in $\mathcal{S}$ (a bijection),
we get $G(f) : G(X) \to G(Y)$ by precomposing a $c : X \to 3$ by <u>inverse</u> of f.
hence define $G(f)(c) = c \circ f^{-1}$.
(check functoriality.) — "inverse" is a contravariant functor!

$\mathrm{inv} : \mathcal{S}^{op} \cong \mathcal{S}$ interesting.

$H(X) = X$ (points)

$$\begin{tikzcd} X \arrow[r, "\sim"] & Y \\ 1 \arrow[u, "p"] \arrow[ur] & \end{tikzcd}$$

$f : X \xrightarrow{\sim} Y$ $\quad H(f) = f$

or $H(f)(p) = f \circ p$.

Puzzles

(1) what's the # partitions of X? <u>Bell</u>

(2) # ways of triangulating regular n-gon? <u>Catalan</u>

10/1
Tuesday

Generating Functions

(d) let
$$0 = \emptyset$$
$$1 = \{0\}$$
$$2 = \{0, 1\}$$
$$n = \{0, \dots, n-1\}$$

(d) let $\mathrm{Set} = [\text{sets, functions}]$
$F = [\text{finite sets, functions}]$
$S = [\text{finite sets, bijections}]$

There are functors
$$S \rightarrowtail F \rightarrowtail \mathrm{Set}.$$

(d) a <u>species</u> is a functor $G : S \to \mathrm{Set}$
a <u>finite</u> species is a functor $G : S \to F$

recall, we can turn a finite set into a number, by taking cardinality $|X|$.

Can we do the same for <u>species</u>?

(d) let $\mathbb{R}[\![x]\!]$ be the set of formal power series

expressions $\sum_{n\geq 0} a_n x^n$ $\quad a_n \in \mathbb{R}$

(possibly with zero radius of convergence)

you can add & multiply these,
and they form a commutative ring.

(d) given a finite species $G: S \to F$,
it generating function $|G| \in \mathbb{R}[\![x]\!]$ is given by

$$|G|(x) = \sum_{n\geq 0} \frac{|G(n)|}{n!} x^n \quad \left(\begin{array}{l} \text{with } n \in S,\ G(n) \in F. \\ \& \ |G(n)| \in \mathbb{N}. \end{array} \right)$$

(x) let C_k be the species of $\boxed{k\text{-colorings}}$

$$C_k(X) = k^X \qquad (\text{with } k = \{0, \dots, k-1\}).$$

then $$|C_k(X)| = \sum_{n\geq 0} \frac{|C_k(n)|}{n!} x^n = \sum_{n\geq 0} \frac{|k^n|}{n!} x^n$$
$$= \sum_{n\geq 0} \frac{k^n}{n!} x^n = \boxed{e^{kx}}$$

(x) let A_k be the species : $\boxed{\text{"being an } k\text{-elmt set"}}$
you can put this structure on an n-elmt set
iff $n = k$. (this is really a property: a true/false structure)

then $|A_k|(x) = \boxed{\frac{1}{k!} x^k}$ (these are like a basis for all formal power series)

$(\Rightarrow |A_1|(x) = x.)$

let's figure out how to <u>add</u> & <u>multiply</u> species.
we can do it for power series, so should correspond:
given $G, H : S \to F$ want $|G+H| = |G| + |H|$

$|G \cdot H| = |G| \cdot |H|$

(d) given $G, H : S \to Set$, let $G+H : S \to Set$ be
$(G+H)(X) = G(X) + H(X)$ ("$G+H$ structure = G structure $\underset{\varepsilon}{\underline{xor}}$ H structure")

(t) $|G+H| = |G| + |H|$ — follow your nose.

(x) $|C_1 + C_2| = |C_1| + |C_2| = e^x + e^{2x}$.

what about $G \cdot H$? ($G H$-structure $\neq$ G structure <u>and</u> H-structure)

$$|G|(x)\,|H|(x) = \Big(\sum_{j\geq 0} \frac{|G(j)|}{j!} x^j\Big)\Big(\sum_{k\geq 0}\frac{|G(k)|}{k!} x^k\Big)$$

$$= \sum_{j\geq 0}\sum_{k\geq 0} \frac{|G(j)||H(k)|}{j!\,k!} x^{j+k}$$

let $n = j+k$ and $j = n-k$ (Day $\underset{\varepsilon}{\underline{\text{convolution}}}$!)

$$\Rightarrow \quad = \sum_{n\geq 0}\sum_{j} \frac{|G(j)||H(n-j)|}{j!\,(n-j)!} x^n$$

$$= \sum_{n\geq 0}\sum_{0\leq j\leq n} \frac{|G(j)\times H(n-j)|}{n!}\frac{n!}{j!(n-j)!} x^n$$

$$= \sum\sum \frac{\binom{n}{j}|G(j)\times H(n-j)|}{n!} x^n$$

$$\overset{\text{want}}{=} \sum_{n\geq 0} \frac{|GH(n)|}{n!} x^n$$

so this works iff

$$|GH(n)| = \sum_{0 \le j \le n} \binom{n}{j} |G(j) \times H(n-j)|$$

and recall $\binom{n}{j} = |\{j\text{-elmt subset of } n\}|$

so it'll work iff "GH-structure =
a subset with G-structure
and complement with H-structure

(x) $C_2 C_1 \cong C_3$ $\quad (C_a C_b = C_{a+b})$
$(\cong C_{2+1} \cong)$ though we haven't defined iso of species

can show $|C_1 C_2| = |C_3|$, because it was by construction

$$e^x \cdot e^{2x} = e^{3x}$$

(Larry was postdoc with Rota...
John only took philosophy classes with him.)

there is a category of species, called Set^S.
{objects : species
morphisms: natural transformations}

$$\begin{tikzcd} S \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$$

$(\forall f : m \to n$ in

(d) family of functions $\{\alpha_n : G(n) \to H(n)\}$ s.t.

$$\begin{tikzcd} G(m) \arrow[r, "\alpha_m"] \arrow[d, "G(f)"'] & H(m) \arrow[d, "H(f)"] \\ G(n) \arrow[r, "\alpha_n"'] & H(n) \end{tikzcd}$$

(continuing in Network Theory)

talking about $S^{op} \cong S$ ...

generating functors? $\hat{G}(X) = \sum_{n \geq 0} G(n) \frac{X^n}{S_n}$ mod out by S_n — make X^n unordered

$\hat{G} : S \to Set$

what is a $\hat{G}$-structure?

sum over n (G-structure on n) $\begin{pmatrix} \text{for an} \\ n\text{-multisubset} \\ \text{of } X \end{pmatrix}$

generalized species, stuff types

problem with cardinalities — quotient by groupoid.

... why are S, F, Set so great?

$S \simeq$ free symmetric monoidal category on 1.

$F \simeq$ free (cocartesian) monoidal category on 1.

let $\underline{S}$ be the skeleton.

$|\cdot| : \underline{S} \to (\mathbb{N}, ids)$ cardinality functor.

(we know tricks with natural numbers that we don't yet know for finite sets.)

F: now we have inclusions,

which give $0 \xrightarrow{!} 1 \xleftarrow{\nabla} 1+1$

(these plus bijections give all functions)

$$\begin{tikzcd} \{1\} \arrow[d, "1"'] \arrow[dr, "\forall \varphi"] \arrow[d, phantom, shift left=4ex, "(\cong)"] & \\ F \arrow[r, dashed, "\exists! \tilde{\varphi}"'] & (C, +) \end{tikzcd}$$

coproduct preserving

how to prove?

also free finite colimits..

(some subcategory of presheaves

(quotienting ~ colimits ?
S_n

and $\mathcal{S}et \cong$ free cat with all (small) colimit on 1

so, it's all about addition.

Q: what's the similar characterization of species?

addition / quotienting

A: free cocompletion of free symmoncat on $\underline{1}$.

this is really the multiplication.

decat: free comm. mon on free comm. mon on x.

$\cong \mathbb{N}[x]$

10/3
Thursday

<u>natural transformations</u>

if $G, H : C \to D$ are functors,
a <u>natural transformation</u> $\alpha : G \to H$
assigns to each $c \in C$ a morphism
$\alpha_c : G(c) \to H(c)$ in D
such that the <u>naturality</u> <u>squares</u>
commute $\forall f : c \to c'$ in C:

$$\begin{array}{ccc} G(c) & \xrightarrow{G(f)} & G(c') \\ \alpha_c \downarrow & \cdot & \downarrow \alpha_{c'} \\ H(c) & \xrightarrow[H(f)]{} & H(c') \end{array}$$

(picture)

(t) if $C, D : Cat$ then $\exists\, D^C : Cat$ where
- objects are functors $G : C \to D$
- morphisms are natural trans $\alpha : G \Rightarrow H$
- composition is $\alpha : G \Rightarrow H$, $\beta : H \Rightarrow J$ gives $\alpha * \beta : G \Rightarrow J$ with $(\alpha * \beta)_c = \alpha_c \circ \beta_c$
- (and obv identities)

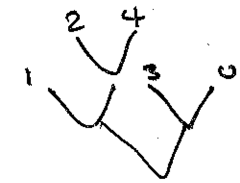
(d) the category of <u>species</u> is Set^S.
the category of <u>finite</u> species is F^S.

(x) let $P : S \to F$ be defined as follows.
$P(X) = \{$bijections of X & $\{e^{2\frac{\pi i}{n}}, \dots, e^{(n-1)\frac{\pi i}{n}}\}$ $(n = |X|)$,
with a triangulation of $\{e^{k\frac{\pi i}{n}}\}_{k=1}^{n-1}$ $\}$

(x) let $B: S \to F$ be defined as follows.

$B(X) = \{$ binary planar rooted trees with leaves labelled by $X \}$, or "X-labelled trees".

an element of $B(5)$

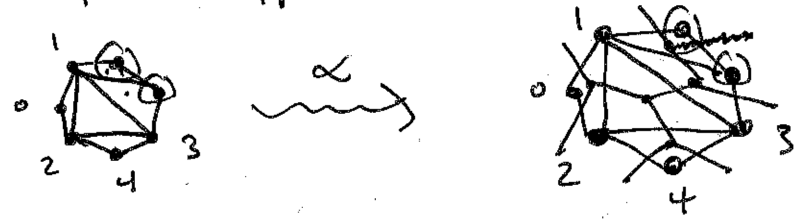


(equivalently, ways of parenthesizing {1,2,3,4,5

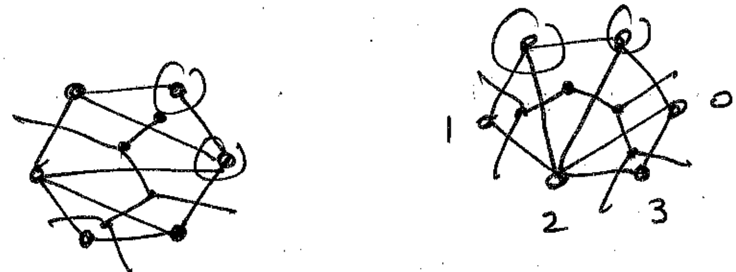
(t) $P \cong B$; ie $\exists$ natural iso $\alpha : P \Rightarrow B$.

(p) need $\alpha_X : P(X) \to B(X)$ that is bijective + natural

... example, suppose $X = 5$. pick an elmt of P(X

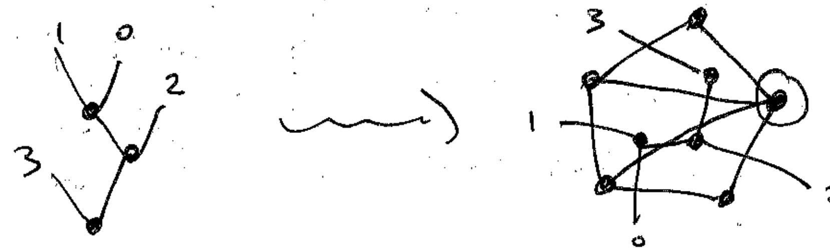


(dual graph)



whoops! we actually need just one special verte
then it works. (label counterclockwise) $\exists$ na

α^{-1}:



now we know $|P(X)| = |B(X)|$, even though we have not computed the numbers!
this is a <u>bijective</u> <u>proof</u>.

(t) if $G, H : S \to F$ then $G \cong H \Rightarrow |G| = |H|$.
(but <u>not</u> conversely.)

Using Generating Functions:

(x) let $TO : S \to F :: \lambda X. \{\text{total orderings of } X\}$.

(then $|TO| = \sum x^n$.)

$TO \cong A_0 + A_1 TO$ (recursive definition).

"to total order a set is to either notice it's a 0-elmt set, or to chop into a one-elmt set, then total order the rest."

c.
ultiple
...ises...