

10/17 Thursday Last time we showed $TO \neq \text{Perm}$.
but $|TO| = |\text{Perm}|$.

So, in what sense is TO "like" Perm ?

a total ordering on $S = \{0, 1, 2, 3, 4\}$

gives a permutation: say $3 < 4 < 0 < 1 < 2$

then we have

	0	1	2	3	4
	↓	↓	↓	↓	↓
	3	4	0	1	2

but this is sneaky because we were using
a predetermined total order on S .

so really we have a map:

$$\alpha: TO \times TO \rightarrow \text{Perm}$$

this is not an isomorphism, but this is:

$$TO \times TO \xrightarrow{\sim} TO \times \text{Perm}$$

$$(x, y) \mapsto (x, \alpha(x, y))$$

so $TO \times TO \cong TO \times \text{Perm}$, and we can use this
to show that $|TO| = |\text{Perm}|$ (can "divide out"
(since $\forall TO(a) \neq 0$)).

new topic \longrightarrow

Derivative of a Species:

Given a species $G: S \rightarrow F$, we want a new species called its derivative $G': S \rightarrow F$ such that

$$|G'(x)| = |G'(x)| \leftarrow \text{(derivative of formal power series)}$$

$$\frac{d}{dx} \left(\sum a_n x^n \right) = \sum n a_n x^{n-1}$$

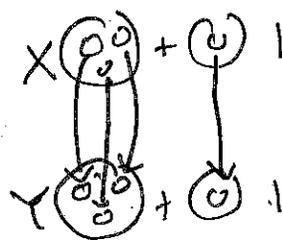
Let $G: S \rightarrow \text{Set}$. Define

$$G'(x) = G(x+1) \quad \forall x \in S$$

and for any bijection $f: X \rightarrow Y$

$$G'(f) = G'(X) \rightarrow G'(Y)$$

$$= G(f + \text{id}_1): G(X+1) \rightarrow G(Y+1)$$



$$\left\{ \begin{array}{l} x \in X \mapsto f(x) \in Y \\ 0 \in 1 \mapsto 0 \in 1 \end{array} \right\}$$

if $G: S \rightarrow F$ then $|G'| = |G|'$.

$$|G'(x)| = \sum_{n \in \mathbb{N}} \frac{|G(n)|}{n!} x^n = \sum_{n \in \mathbb{N}} \frac{|G(n+1)|}{n!} x^n$$

$$= \sum_{n \geq 1} \frac{|G(n)|}{(n-1)!} x^{n-1} = \frac{d}{dx} \sum_{n \geq 0} \frac{|G(n)|}{n!} x^n = |G'(x)|.$$

k -colorings: $C_k: S \rightarrow F$, $C_k(x) = k^x = S(x, k)$.

as we saw $|C_k|(x) = \sum_{n \in \mathbb{N}} \frac{k^n}{n!} x^n = \sum_{n \in \mathbb{N}} \frac{k^n}{n!} x^n = e^{kx}$.

Let's call C_k " E^{kx} ", so $|E^{kx}| = e^{kx}$. (well... maybe not.)

since $\frac{d}{dx} e^{kx} = k e^{kx}$, we could hope that

$$\frac{d}{dx} C_k \approx k C_k$$

what does this mean? $\left(\sum_{k=1}^{\infty} C_k \right)$
or equivalently, $k \times C_k(-)$.

Ⓟ it's true.

$$\hookrightarrow \frac{d}{dx} C_k(x) = C_k'(x) = C_k(x+1) = k^{x+1} F(x+1, k) \\ \approx k^x \times k^1 \approx k \times C_k$$

so we get $|C_k'| = k C_k$

$$\frac{d}{dx} e^{kx} = k e^{kx}$$

this class is about seeing numbers as a shadow of finite sets, through categorification. to recover $\mathbb{R} \setminus \mathbb{N}$, there are things with weird cardinalities... (groupoids) but still don't know what to do about negatives

Ⓟ let $C_{\text{cyc}}: S \rightarrow F$ be the species of cyclic orderings:

$\begin{array}{c} \circ \rightarrow 1 \\ \downarrow 2 \\ \uparrow 3 \leftarrow \end{array}$
 a cyclic ordering on a finite set X is a bijection $f: X \rightarrow X$ (permutation) with one cycle.

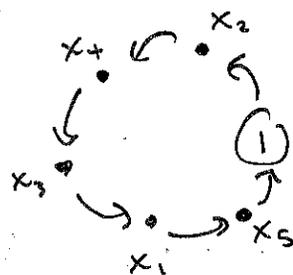
(a cycle of f is a set of the form $\{f^n(x)\}_{n=0}^{\infty}$)
(every permutation is a disjoint union of cycles.)

let $\text{Cyc}(X) = \{\text{cyclic orderings on } X\}$.

what is $|\text{Cyc}(n)|$?

note: * $\text{Cyc}' \cong \text{TO}$. *

$\text{Cyc}'(X) = \text{Cyc}(X+1) \ni$
(start with $f(0)$, to $f^{n+1}(0)$.)



$$x_2 < x_4 < x_3 < x_1 < x_5$$

so, we must have that

$$|\text{Cyc}'| = |\text{TO}| = \sum_{n \in \mathbb{N}} \frac{n!}{n!} x^n = \frac{1}{1-x}$$

$$\Rightarrow |\text{Cyc}'(x) = \int \frac{dx}{1-x} = -\ln(1-x) + C$$

but, probably just easier to do it termwise.

$$\sum \frac{|\text{Cyc}'(n)|}{n!} x^n = \left(\sum_{n \in \mathbb{N}} \frac{x^{n+1}}{n+1} \right) + C, \quad \text{so for } n > 0$$

$$\frac{|\text{Cyc}'(n)|}{n!} = \frac{1}{n} \Rightarrow |\text{Cyc}'(n)| = (n-1)! \quad \square$$

and $|\text{Cyc}'(0)| = C \rightarrow C = 0$.

so, we have $|\text{Cyc}'(x) = \sum_{n \in \mathbb{N}} \frac{x^n}{n!} = -\ln(1-x)$.

(*) $(FG)' \cong F'G + FG'$ (beautiful) *

$$\hookrightarrow (FG)'(x) = (FG)'(x+1) = \sum_{Y \in X+1} F(Y) \times G(x+1-Y)$$

$$\cong \sum_{Y \in X} F(Y) \times G(x-Y+1) + \sum_{1 \in Y \in X} F(z+1) \times G(x-z) = \underbrace{F'G(x)}_{\text{}} + \underbrace{FG'(x)}_{\text{}}$$