

11/19 (t) (contd.) $X: \text{Gpd}$, $Y: \text{Cat}_{\oplus} \leftarrow$ coproducts

then $H: X \rightarrow Y$ is indecomposable (as $\text{obj} \in Y^X$)

iff: 1) H is supported on one component of X
($H \cong Y \leftrightarrow X \leftrightarrow X_\alpha \leftrightarrow \text{sk}(X_\alpha) = G_\alpha$)
and 2) each $H_\alpha: G_\alpha \rightarrow Y$ is indecomposable.

(if $Y = \text{Set}$, then every indecomposable $G_\alpha \rightarrow Y = \text{Set}$
is isomorphic to an action of G_α on G_α/K for some $K \subseteq G_\alpha$.)

"it's not dumb to study groupoids
—more than just 'a bunch of disjoint groups'
because of monoidal structures.
that's how we get structures on species."

Felix Klein started the Erlangen program:
studying geometrical figures as invariant
subgroups of symmetry-actions.

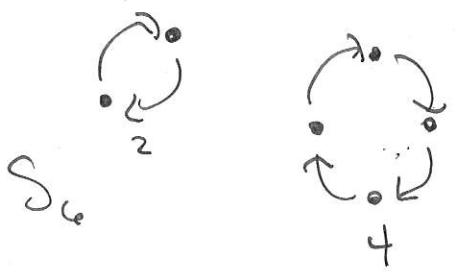
"we're sneaking over to studying this..."

(*) $\textcircled{1} X = S$, $Y = \text{Set} \Rightarrow Y^X = \text{species}$

$\textcircled{2} X = S$, $Y = \text{Vect} \Rightarrow$ "linear" species

for #1, indecomps \leftrightarrow subgroups of S_n 's (hard)

#2, indecomps \leftrightarrow conjugacy classes of permutations
(easier — determined by cycle structure)



4+2 labels a conjugacy class in S_4



Young diagrams

Stirling Numbers

Binomial coefficient

$$\binom{n}{k} = |\{ \text{kelement subsets of } n \}|$$

stirling
first kind

$$[n]_k = |\{ \text{permutations of } n \text{ with } k \text{ cycles} \}|$$

Stirling
second kind

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = |\{ \text{partitions of } n \text{ with } k \text{ parts} \}|$$

Let's do the second kind.

there's a species here ... $|\text{Part}_k(n)| = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$.

$\text{Part}_k = \text{"being a finite set partitioned into } k \text{ parts"}$

$(\text{Part} \cong \sum_{k \in \mathbb{N}} \text{Part}_k \quad \text{— bad pun.})$

$$e^{x-1} = |\text{Part}(x) = \sum_{k \in \mathbb{N}} |\text{Part}_k|(x) = \sum_{k \in \mathbb{N}} \left(\sum_{n \in \mathbb{N}} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{x^n}{n!} \right)$$

$$\text{so: } \text{Part}_k \cong A_k \circ NE \cong (NE)^k.$$

$$\Rightarrow |\text{Part}_k| = |A_k| \circ |NE| = \left(\frac{x^k}{k!} \right) \circ (e^x - 1)$$

$$\text{so we have } \sum_{n \in \mathbb{N}} \frac{\{n\}}{n!} x^n = \frac{(e^x - 1)^k}{k!}$$

$$(\text{so also, } \{n\} = \frac{d^n}{dx^n} \left. \frac{(e^x - 1)^k}{k!} \right|_{x=0}).$$

→ not necessarily the easiest way to compute,
but may lead to lots of good proofs.

Q: what is $\sum_{k=0}^n \{k\}$? #partitions of n ,
or n th "Bell number".

$$\text{take } \text{Part}_k \cong A_k \circ \text{NE}$$

and take derivative of both sides

$$\text{Part}'_k \cong (A'_k \circ \text{NE}) \cdot \text{NE}'$$

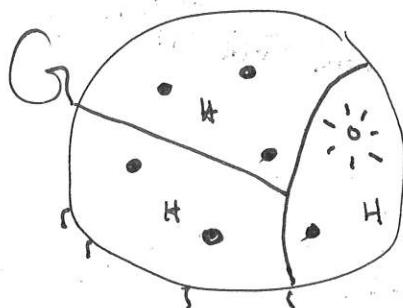
[do later: prove chain rule.] $(G \circ H)' \cong (G' \circ H) \cdot H'$
 $(G \circ H)(x+1) \cong$

$$\cong (A'_{k-1} \circ \text{NE}) \cdot \text{Exp}$$

so

$$\text{Part}'_k \cong \text{Part}'_{k-1} \cdot \text{Exp}$$

to partition $X+1$ into k parts \leftrightarrow to chop into two parts,
partition first in $k-1$ parts.



$$\{n+1\}_k = \sum_{i=0}^n \binom{n}{i} \{i\}_{k-1}$$

Seminar Stuff Types & Groupoid Cardinality

(d) stuff type: pseudofunctor $G: S \rightarrow \text{Gpd}$

$$\phi_{f,g}: G(f) \circ G(g) \xrightarrow{\sim} G(f \circ g)$$

$G(X)$ is "the groupoid of G -stuff on X ".

doing the Grothendieck construction,

get a groupoid $\int G$ "finite sets with G -stuff"

$$\begin{matrix} P \downarrow & \text{"forgets the } G\text{-stuff"} \\ G \end{matrix}$$

⊗ if $G: S \rightarrow \text{Set} \rightarrow \text{Gpd} :: 2X \cdot 2^X$,

$$\text{then } \int G = \left[\begin{smallmatrix} \text{2-colored} \\ \text{finite sets, bijections} \\ \text{such that } \forall i \end{smallmatrix} \right] \xrightarrow{\cong} Y$$

$$(\int [E, c] \cong \mathcal{Y}_c)$$

but this was a speccer; what about a full stuff type?

we could replace 2 by a groupoid,

$$\text{e.g. } \mathbb{Z}_2 = \mathbb{C} \circ \mathbb{D}^{-1}$$

this will be thought of as a " $\frac{1}{2}$ -colored set".

$$G(X) = (\mathbb{Z}_2)^X = \prod_{x \in X} \mathbb{Z}_2 : \text{Gpd}.$$

"whereas lots of 2-colorings and not many morphisms,
there's now one $\frac{1}{2}$ -coloring, and lots of morphisms"?

$$\int G = \left[\begin{smallmatrix} \text{finite sets } X \\ \text{with } f: X \rightarrow \mathbb{Z}_2 \\ \text{(just one)} \end{smallmatrix} \right] \xrightarrow{\text{bijections}} \left[\begin{smallmatrix} X \xrightarrow{f} X' \\ \text{ie a map } f \\ \text{giving } \frac{x \in X}{x \in X'} \xrightarrow{\cong} \mathbb{Z}_2 \end{smallmatrix} \right]$$

Given a stuff type $G: S \rightarrow \text{Gpd}$,
 its generating function $|G| \in \mathbb{R}[[x]]$
 if $\forall n \quad G(n) \in \text{Gpd}$ is finite. ($G: S \rightarrow \text{FinGpd}$)

Let $p: \int G \rightarrow S$, then ~~rather~~ \mathbb{N}^*

$$|G|(x) = \sum_{n \in \mathbb{N}} |\mathcal{P}^{-1}(n)| x^n. \quad (\text{generalizing cardinality})$$

Here $\mathcal{P}^{-1}(n)$: Gpd is the ^(full) essential preimage of n :
 the category with objs $x \in \int G$ s.t. $p(x) \cong n$,
 and all morphisms between those.

also, if $A: \text{Gpd}$, let $\underline{A} = \{\text{isoclasses in } A\}$
 and define

$$|A| = \sum_{\substack{x \in A \\ x = [a]}} \frac{1}{|\text{Aut}(a)|} \quad \star \quad \star$$

$$\sum \frac{1}{|\text{Aut}(a)|}$$

$$(*) |Z_2| = \frac{1}{2}$$

$$\left| \begin{array}{c} \bullet & \circ \\ \downarrow f & \downarrow g \\ \bullet & \circ \end{array} \right| = 1 \quad \left(\begin{array}{l} \text{one isoclass,} \\ \text{but iso in a} \\ \text{unique way} \end{array} \right) \quad \leftarrow \begin{array}{l} \text{this one is} \\ \text{"codiscrete"} \\ \text{equivalent to} \\ \text{the terminal cat.} \end{array}$$

$$\left| \begin{array}{c} \bullet & \circ \\ \downarrow f & \downarrow g \\ \bullet & \circ \\ \downarrow f^{-1} & \downarrow g^{-1} \\ \bullet & \circ \end{array} \right| = \frac{1}{2} \quad \left(\begin{array}{l} Z_2 \text{ is a skeleton,} \\ \text{and } Z_2 \cong B \Rightarrow |Z_2| = |B| \end{array} \right)$$

$$(*) |G|(x) = 2^x, \text{ then } |G|(x) = \sum |\mathcal{P}^{-1}(n)| x^n$$

where $\mathcal{P}^{-1}(n)$ is the groupoid of 2-colored n -chrom sets.

$n=2$



1)

Skeleton: $\text{IC} \rightarrow \boxed{\textcircled{0} \quad \textcircled{0}} \xrightarrow{\text{swap}}$



$$\rightarrow |\rho^{-1}(2)| = \frac{1}{2} + 1 + \frac{1}{2} = 2$$

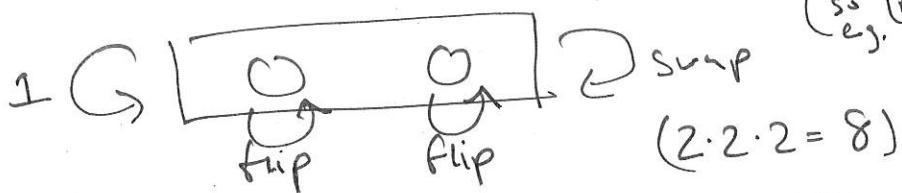
in fact, $|\rho^{-1}(n)| = \frac{2^n}{n!}$ — (did this for species,
but didn't motivate $\frac{1}{n!}$)

$$\text{so } |G|(x) = \sum \frac{2^n}{n!} x^n = e^{2x}.$$

(*) if $G(x) = \mathbb{Z}_2^X$, "½-colorings"

$$\text{we claim: } |G|(x) = e^{\frac{1}{2}x} = \sum \frac{(1/2)^n}{n!} x^n \quad *$$

$$(\text{so, } |\rho^{-1}(2)| = \frac{1}{8})$$



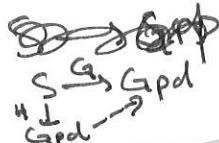
* and universal
of $[G, H]$'s

still have all operations or stuff types!

$$G, H: S \rightarrow \text{Gpd} \rightsquigarrow G \circ H: S \rightarrow \text{Gpd} \quad (|G \circ H| = |G| \circ |H|)$$

if we take H to be constant at $x \in \text{Gpd}$,

then $G \circ H$ could be called " $G(x)$ ", evaluation.



Stirling Numbers, contd.

11/21
Thursday

$$\binom{n}{k} = \left| \{ \text{k-elt subsets of } n \} \right| \quad \begin{array}{l} \text{"Pascal" discovered} \\ \text{in China} \end{array}$$

1	1	1	1
1	2	1	2
1	3	3	1

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \begin{array}{l} \text{(the k-elt subset either} \\ \text{does or does not contain "+1"}) \end{array}$$

recurrence relation → from a natural isomorphism. These have analogues for Stirling numbers.

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \left| \{ \text{permutations of } n \text{ with } k \text{-cycles} \} \right|$$

$$\left[\begin{matrix} n+1 \\ k \end{matrix} \right] = n \left[\begin{matrix} n \\ k \end{matrix} \right] + \left[\begin{matrix} n \\ k-1 \end{matrix} \right] \quad \begin{array}{l} \text{(n choices of where to map "+1";} \\ \text{or be its own cycle)} \end{array}$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left| \{ \text{partitions of } n \text{ into } k \text{ parts} \} \right|$$

$$\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\} \quad \begin{array}{l} \text{(k choices to put "+1" in)} \\ \text{or be its own part)} \end{array}$$

we saw $\text{Part}_k \cong A_k \circ \text{NE}$ and $|\text{Part}_k(n)| = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$

$$\Rightarrow \text{Part}_k' \cong (A_k \circ \text{NE}) \cdot \text{Exp} \cong \text{Part}_{k+1} \cdot \text{Exp}$$

$$\begin{aligned} \Rightarrow \text{Part}_k'(n) &\cong \text{Part}_k(n+1) \cong (\text{Part}_{k+1} \text{Exp})(n) \\ &= \sum_{Y \in n} \text{Part}_{k+1}(Y) * \text{Exp}(nY) \end{aligned}$$

$$\dots \Rightarrow \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} = \sum_{i=k}^n \binom{n}{i} \left\{ \begin{matrix} i \\ k \end{matrix} \right\}$$

$$\begin{aligned}
 \text{but also, } \text{Part}_k' &\cong \text{Part}_{k-1} \cdot \text{Exp} \\
 &\cong \text{Part}_{k-1} \cdot \left(\sum_{n=0}^{\infty} A_n \right) \\
 &\cong \text{Part}_{k-1} \cdot (A_0 + \text{NE}) \\
 &\cong \text{Part}_{k-1} \cdot A_0 + \text{Part}_{k-1} \cdot \text{NE}
 \end{aligned}$$

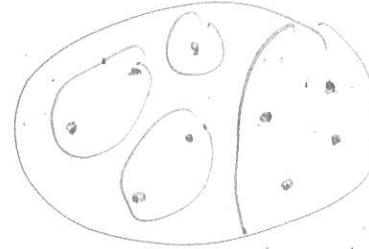
but note that if G_k is any species,

$$G_k \cdot A_0 \cong G_k. \quad (A_0 \text{ is multiplicative identity})$$

$$\Rightarrow \cong \text{Part}_{k-1} + \text{Part}_{k-1} \cdot \text{NE}$$

next -- it looks like

$$\text{Part}_{k-1} \cdot \text{NE} \cong \text{Part}_k$$



but it's really

$$\text{Part}_{k-1} \cdot \text{NE} \cong \text{Pt} \circ \text{Part}_k$$

(want to demonstrate why $|\text{Part}_{k-1} \cdot \text{NE}(x)| = k |\text{Part}_k(x)|$).

\Rightarrow coefficients are equal, so

$$\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\}$$

So we have a new triangle ...

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	row sums
$n=0$	1					1
$n=1$	0	1				1
$n=2$	0	0	1			2
$n=3$	0	1	0	1		5
$n=4$	0	1	3	1	1	15

(Bell numbers)

Stirling Numbers of the First Kind

let $\text{Perm}_k(X) = \{\text{permutations of } X \text{ with } k \text{ cycles}\}$

$$\text{so } |\text{Perm}_k(n)| = \begin{bmatrix} n \\ k \end{bmatrix}.$$

now $\text{Perm}_k \cong A_k \circ \text{Cyc}$.

we found that $|\text{Cyc}(x)| = \ln \frac{1}{1-x}$,

$$\begin{aligned} &(\text{we saw that}) \\ &\text{Perm} \cong \text{Exp} \circ \text{Cyc} \\ &\cong (\sum A_k) \circ \text{Cyc} \\ &\cong \sum A_k \circ \text{Cyc} \\ &\cong \sum \text{Perm}_k \end{aligned}$$

$$\begin{aligned} \text{so } |\text{Perm}_k|(x) &= |A_k| \left(\ln \frac{1}{1-x} \right) \\ &= \frac{\ln \left(\frac{1}{1-x} \right)^k}{k!} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} n \\ k \end{bmatrix} = \frac{d^n}{dx^n} \left. \frac{\ln \left(\frac{1}{1-x} \right)^k}{k!} \right|_{x=0} \quad \begin{aligned} &(\text{and so on,}) \\ &(\text{as for the second bird.}) \end{aligned}$$

in the end we get $\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$.