For $\lambda_i \neq 0$, let $\lambda_i^V$ be the dual basis of $\lambda_i$.

**(1)** Let $\lambda^V = \mathbb{C}_2 \ominus e_1, e_2$

and

$$\bigwedge^* C = \bigwedge \bigwedge \bigwedge$$

s.t.

$$\bigwedge C = \bigwedge$$

and s.t. if

$$X = A I + A^{-1} U$$

Then the inverse

$$X = (X)^{-1} \text{ is } A^{-1} U + A I$$

*Solution: (when $A = 1$)*

$$\bigwedge e_1 \otimes e_1 \bigwedge e_1 \otimes e_2 \bigwedge e_2 \otimes e_1 \bigwedge e_2 \otimes e_2$$

$$\begin{align*}
\bigwedge &= 0 \\
\bigwedge &= 1 \\
\bigwedge &= -1 \\
\bigwedge &= 0
\end{align*}$$
so guess: for other $A$:

\[
\begin{array}{cccc}
    e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_1 & e_2 \otimes e_2 \\
    \downarrow & \downarrow & \downarrow & \downarrow \\
    0 & A & -A^{-1} & 0
\end{array}
\]

Then:

\[
\begin{array}{c}
    v \otimes v^n \Longrightarrow e_1 \otimes e_2 - e_2 \otimes e_1 \\
    (\text{when } A = 1)
\end{array}
\]

guess: \[1 \mapsto -A e_1 \otimes e_2 + A^{-1} e_2 \otimes e_1.\]

Now check what happens when we send in $e_2$.

\[
\begin{array}{c}
    e_2 \downarrow \\
    -A e_2 \otimes e_1 \otimes e_2 + A^{-1} e_2 \otimes e_2 \otimes e_1 \\
    \downarrow \\
    -A (-A^{-1}) e_2 \\
    \downarrow \\
    e_2 \checkmark
\end{array}
\]

so \[1 \mapsto \checkmark\]
Now check $1 = \nabla$

$$
e_1, \downarrow \quad -A e_1 \otimes e_2 \otimes e_1 + A^{-1} e_2 \otimes e_1 \otimes e_1$$

$$
- A e_1 \otimes (-A^{-1}) + O
\downarrow e_1 \checkmark$$

$$
e_2, \downarrow
-A e_1 \otimes e_2 \otimes e_2 + A^{-1} e_2 \otimes e_1 \otimes e_2$$

$$
O \quad \quad A
\downarrow e_2 \checkmark$$

Now see what $\cup$ does to $\cup$

$$
e_1 \otimes e_1 \quad e_1 \otimes e_2 \quad e_2 \otimes e_1 \quad e_2 \otimes e_2
\downarrow 0 \quad \downarrow A \quad \downarrow -A^{-1} \quad \downarrow 0$$

$$
\downarrow -A^2 e_1 \otimes e_2 + e_2 \otimes e_1
\downarrow 0
\downarrow e_1 \otimes e_2 - A^2 e_2 \otimes e_1$$
\[ X = A^{-1} + A U \]

\[ X = A^{-1} + A^{-1} U \]

What does this do to the basis vectors:

- \( e_1 \otimes e_1 \)
- \( e_1 \otimes e_2 \)
- \( e_2 \otimes e_1 \)
- \( e_2 \otimes e_2 \)

- \( A e_1 \otimes e_1 \)
- \( A e_1 \otimes e_2 \)
- \( A e_2 \otimes e_1 \)
- \( A e_2 \otimes e_2 \)

- \( -A e_1 \otimes e_2 + A^{-1} e_2 \otimes e_1 + A^{-1} e_1 \otimes e_2 \)
- \( -A^{-3} e_2 \otimes e_1 \)

\[ A^{-1} e_2 \otimes e_1 \]

Note: when \( A = I \) reduces to correctly

Switching 2 vectors

Ex) \( e_1 \otimes e_2 \rightleftharpoons e_2 \otimes e_1 \)

We could just check that

\[ \mathcal{L} = \mathcal{X} \]
Now check \( \mathcal{U} \) does to

\[
\begin{align*}
\text{e}_1 \otimes \text{e}_1 & \quad \text{e}_1 \otimes \text{e}_2 & \quad \text{e}_2 \otimes \text{e}_1 & \quad \text{e}_2 \otimes \text{e}_2 \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
0 & \quad A & \quad -A^{-1} & \quad 0 \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
-\text{A}^2 \text{e}_1 \otimes \text{e}_2 + \text{e}_2 \otimes \text{e}_1 & \quad \text{e}_1 \otimes \text{e}_2 - \text{A}^2 \text{e}_2 \otimes \text{e}_1 & \quad 0 \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
0 & \quad 0 & \quad 0 & \quad 0
\end{align*}
\]
\[
\begin{align*}
X &= A^{-1} U + A U \\
X &= A^{-1} U + A U
\end{align*}
\]

What does this do to the basis vectors:

\[
\begin{align*}
e_1 \otimes e_1 & \quad e_1 \otimes e_2 \\
e_2 \otimes e_1 & \quad e_2 \otimes e_2
\end{align*}
\]

\[
\begin{align*}
Ae_1 \otimes e_1 & \quad Ae_1 \otimes e_2 \\
Ae_2 \otimes e_1 & \quad Ae_2 \otimes e_2
\end{align*}
\]

\[
\begin{align*}
-Ae_1 \otimes e_2 + A^{-1} e_2 \otimes e_1 & \quad + A^{-1} e_1 \otimes e_2 \\
\| & \quad - A^{-3} e_2 \otimes e_1
\end{align*}
\]

\[
A^{-1} e_2 \otimes e_1
\]

Note: when \((A = 1)\) reduces to correctly

Switching 2 vectors

ex) \(e_1 \otimes e_2 \rightarrow e_2 \otimes e_1\)

We could just check that \(\| = \langle X, V \rangle\).
\[ X = A + A^{-1} U \]

\[ = 1 + A^2 U + A^{-2} U + U \]

The last 3 will cancel if:

\[ O = -(A^2 + A^{-2}) \]

\[ \begin{array}{c}
\circ \\
\downarrow \\
-Ae_1 \otimes e_2 + A^{-1} e_2 \otimes e_1
\end{array} \]

\[ \downarrow \cup \text{up} \]

\[-A^2 - A^{-2} \checkmark.\]

Lin Alg Fact: \( ST = I \) iff \( TS = I \)

if \( S, T: L \rightarrow L \) are linear

and \( L \) is finite dim'd.
New material: (no longer HW #1)

All this stuff is secretly the study of the “spin-1/2 representation” of the “quantum group” called “quantum $SL(2, \mathbb{C})$.”

And when $A = 1$, this reduces to the group $SL(2, \mathbb{C})$—all transformations preserving the symplectic structure on $\mathbb{C}^2$.

Note: A quantum group is not a group!

It’s an algebra (v-space w/ annc. bilinear product & unit) that pretends to be a group.

(Last time—we showed how a group pretends to be an algebra.)

An example is the group algebra $C[G]$ of a group: finite linear comb. of elts. of $G$.

\[ (\sum a_i g_i)(\sum b_j h_j) = \sum a_i b_j g_i h_j \]

Quantum groups are like group algebras.

Defn: A representation of an algebra $A$ on a v-space $V$ is a linear map

\[ \rho: A \rightarrow \text{End}(V) \quad \text{s.t.} \]

\[ \rho(aa') = \rho(a)\rho(a') \quad \text{and} \quad \rho(1) = 1_V. \]
Prop: A representation of the group $G$ is the same thing as a rep. of the algebra $\mathbb{C}[G]$.

(Let the two be in 1-1 correspondence).

Proof: Given a rep. $\rho$ of $G$, let $\hat{\rho}$ be a rep. of $\mathbb{C}[G]$ by:

$$\hat{\rho}(\sum a_i g_i) = \sum a_i \rho(g_i)$$

Given a representation $\rho$ of $\mathbb{C}[G]$, let $\hat{\rho}$ be a rep. of $G$ by:

$$\hat{\rho}(g) = \rho(g). \quad G \subseteq \mathbb{C}[G].$$

Given two reps. of $G$, say

$$\rho: G \rightarrow \text{End}(V)$$

$$\rho': G \rightarrow \text{End}(V')$$

we get a rep.,

$$\rho \circ \rho': G \rightarrow \text{End}(V \otimes V')$$

by

$$(\rho \circ \rho')(g) = \rho(g) \otimes \rho'(g)$$

We'll check that this $\rho \circ \rho'$ is a rep.

We can't do this trick to tensor reps. of algebras:

$$(\rho \circ \rho')(a) = \rho(a) \otimes \rho'(a).$$

Because: Need $\rho \circ \rho': A \rightarrow \text{End}(V)$ linear, but it's NOT!
proof that \((\rho \otimes \rho')\) isn't linear:

\[ (\rho \otimes \rho')(2a) = \rho(2a) \otimes \rho'(2a) \]

\[ = 4 \rho(a) \otimes \rho'(a) \]

We can try to deduce \(\rho \otimes \rho'\) were linear,

\[ (\rho \otimes \rho')(2a) = 2 \rho(a) \otimes \rho'(a). \]

We should be able to tensor reps. of group algebras, however.

In fact - we **can** tensor reps. of a group algebra from what we've already said about group algs.

Given \(\rho: \mathbb{C}[G] \rightarrow \text{End}(V)\)

\(\rho': \mathbb{C}[G] \rightarrow \text{End}(V')\) we get:

\[(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g) \quad g \in \text{group } G.\]

So

\[(\rho \otimes \rho')(\sum_{i} g_{i}) = \sum_{i} \rho(g_{i}) \otimes \rho'(g_{i}) \]

since \((\rho \otimes \rho')\) as a rep. of an alg. must be linear.

Trick - a group alg. has as basis the elts. of the group.

This "duplication" of group elements gives us a map:
"duplication" map:

\[ \Delta : C[G] \rightarrow C[G] \otimes C[G] \]

\[ g \mapsto g \otimes g \]

where \( g \in G \) (the group) and \( g \) is a basis elt. for \( C[G] \) (elts of the group form a basis of the group alg).

with \( \rho \otimes \rho' : C[G] \rightarrow \text{End}(V \otimes V) \)

given by:

\[ \begin{array}{ccc}
C[G] & \xrightarrow{\Delta} & C[G] \otimes C[G] \\
\downarrow \rho \otimes \rho' & & \downarrow \rho \otimes \rho' \\
\text{End}(V) \otimes \text{End}(V') & \xrightarrow{1 \otimes 1} & \text{End}(V \otimes V') \\
\end{array} \]

\[ g \mapsto g \otimes g \xrightarrow{\rho \otimes \rho'} \rho(g) \otimes \rho'(g) \]
Suppose $A$ is an algebra, $\Delta: A \rightarrow A \otimes A$ is linear and 
\[ \rho: A \rightarrow \text{End}(V), \quad \rho': A \rightarrow \text{End}(V'). \]

We can try to define a rep. $\rho \otimes \rho': A \rightarrow \text{End}(V \otimes V')$, by the same formula:

\[
(*) \quad A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho \otimes \rho'} \text{End}(V) \otimes \text{End}(V') \cong \text{End}(V \otimes V').
\]

**BUT:** Is it a rep.?

Is $(\rho \otimes \rho')(aa') = (\rho \otimes \rho')(a)(\rho \otimes \rho')(a')$ and

\[
(\rho \otimes \rho')(1) = 1_{V \otimes V'} ??
\]

Let's use pictures to represent representations:

A rep. $\rho: A \rightarrow \text{End}(V)$ looks like:

\[
A \xrightarrow{\rho} V \xrightarrow{\lambda} A \otimes V
\]

\[\tilde{\rho}(\cdot)(\cdot) \text{ gives } \rho: A \rightarrow \text{End}(V)\]

\[
\rho(a)(v) = \tilde{\rho}(a \otimes v)
\]

**Must satisfy:**

\[
A \xrightarrow{\rho} V \xrightarrow{m} A \otimes V \xrightarrow{\rho} V
\]

where $m$ is mult.

\[m: A \otimes A \rightarrow A\]

is mult. in $A$. 
Given reps \( \rho: A \rightarrow \text{End}(V) \)
\( \rho': A \rightarrow \text{End}(V') \)

what does \( (\rho \otimes \rho'): A \rightarrow \text{End}(V \otimes V') \) look like?

we draw

\[ \begin{array}{ccc}
A & \rightarrow & V \otimes V' \\
\rho \otimes \rho' & = & A \\
\end{array} \]

see (*) we duplicate

prev pg then break up.

We draw \( \Delta: A \rightarrow A \otimes A \) as

\[ \begin{array}{ccc}
A & \rightarrow & A \\
\Delta & \rightarrow & A \otimes A \\
\end{array} \]
Now — Is \((\rho \otimes \rho')\) a rep.? For this to be true, we need:

1. \((\rho \otimes \rho')(aa') = (\rho \otimes \rho')(a)(\rho \otimes \rho')(a')\) and 
   
2. \((\rho \otimes \rho')(1) = 1_{V \otimes V'}\)

(\(\circ\) Says:

\[\begin{array}{c}
\circ \\
\alpha \\
\alpha' \\
\odot V \otimes V'
\end{array}\]

We want this to be equal

LHS:

We know \(\rho\) is a rep:

ie)

\[\begin{array}{c}
\rho \\
\rho'
\end{array}\]

and similarly for \(\rho'\) (is a rep).
LHS: So we use the fact that $p$ is a rep to redraw as:

```
\[ \ldots \]
```

II

Since $p'$ is also a rep so we can do the same thing:

```
\[ \ldots \]
```

Both the LHS & RHS will be equal if:

```
\[ \ldots \]
```
So, let's assume this as one of the axioms for a bialgebra (e.g., quantum group). We also need: \((\rho \otimes \rho')(1) = 1 \otimes 1\).

\[\begin{align*}
1 & \rightarrow \nu \otimes \nu' \\
\rho \otimes \rho' & = & \nu \otimes \nu'
\end{align*}\]

Recall:

\[\begin{align*}
1 & \rightarrow \nu \\
\rho & = & \nu
\end{align*}\]

And similarly for \(\rho'\).

So, the eqn. will hold if:

\[\begin{align*}
1 & \otimes 1 \\
\Delta & = & 1
\end{align*}\]

So we also assume this in defn. of bialgebra.
We'd like to have:

\[(\rho \otimes \rho') \otimes \rho'' = \rho \otimes (\rho' \otimes \rho'')\]

(This is true for reps of groups:

\[\left[(\rho \otimes \rho') \otimes \rho''\right](g) = \rho(g) \otimes \rho'(g) \otimes \rho''(g)\]

\[\left[\rho \otimes (\rho' \otimes \rho'')\right](g)\]

and thus true for reps. of group algebras.)

Want:

\[\begin{array}{ccc}
\rho \otimes (\rho' \otimes \rho'') & \sim & (\rho \otimes \rho') \otimes \rho'' \\
\end{array}\]
The dual of anything is upside-down version

There will be equal if our alg. is assoc. (well-upside-down assoc.)

Note: \( \Delta \) is "upside-down" muet.

\( \mu_{et} \) - puts things together
\( \Delta \) - tears things apart.

The two on prev. pg will be equal if:

\[
\Delta \times \Delta = \Delta \\
\text{upside-down comultiplicativity.}
\]

So - call \( \Delta : A \rightarrow A \otimes A \) comultiplication
and call this law

\[
\Delta = \Delta \\
\text{coassociativity.}
\]

While we're at it, let's assume \( A \) has a counit,

\( e : A \rightarrow C \)

we can think of \( e \) as an elt of the alg. or a map from \( A \rightarrow C \).

we draw as:

\[
\text{satisfying left/right counit laws:}
\]

\[
\Delta = \Delta \\
\text{e} = e
\]
Defn: A vector space $W$ with a coassoc., comult. $\epsilon$, counit satisfying the left and right counit laws is called a coalgebra.

Exs) of coalgebras:

1. If $A$ is an algebra, we can form a coalg. by taking its dual.

* turns everything upside down

A has assoc. mult. $m: A \otimes A \to A$
and unit $i: C \to A$ satisfying left & right unit laws so $A^*$
will become a coalgebra w/

$$\Delta = m^*: A^* \to A^* \otimes A^*$$

Taking duals we can reverse all linear maps! and $e = i^*: A^* \to C$

Defn: We define a bialgebra $(A, m, i, \Delta, e)$
to be a vector space $W$ s.t. $(A, m, i)$ is an algebra,
$(A, \Delta, e)$ is a coalgebra such that:
comult. is an alg. hom.

\[ \begin{array}{c}
\xymatrix{
M & \ar[r]^{\Delta} & M \ar[d]^{m} \\
\Delta \ar[u]^{m} & & \Delta \\
\ar[u] & & \ar[u]}
\end{array} \]

and

Thm: If \( A \) is a bialgebra and \( \rho, \rho' \) are reps of \( A \) as an algebra, then we can define a rep \( \rho \otimes \rho' \) of the algebra \( A \) via:

\[ A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho \otimes \rho'} \text{End}(V) \otimes \text{End}(V') \cong \text{End}(V \otimes V') \]

and we have \( (\rho \otimes \rho') \otimes \rho'' = \rho \otimes (\rho' \otimes \rho'') \)

Fact - quantum groups will be bialgebras.
HW#2: Using \( X = A \| I + A^{-1} \| U \)

\( O = -(A^2 + A^{-2}) \)

calculate numbers for

Note that the fund group can't tell that these 2 knots apart.

2 versions of trefoil.

Show that the results are different, and you can't make them the same by multiplying by any power of \(-A^3\).

\( O = -A^3 \)

Conclusion: These are different knots.

The number we get by this recipe from any knot (or link) \( K \) is called the Kauffman bracket, and denoted: \( \langle K \rangle \).
Any compact group $G$ has a unique measure $\mu$ which has

$$\int_G \mu = 1$$

and is invariant under left and right translations and inversion.

This lets us define $L^2(G) = L^2(G, \mu)$.

**Question:** Can we find a nice orthonormal basis of $L^2(G)$? Answer will involve reps. of $G$.

**Ex)** When $G$ is finite - use $\delta_{ij}$ (Kronecker delta).

**Ex)** $G = U(1) = \{ e^{i\theta} | \theta \in \mathbb{R} \}$ unit complex #’s is a group under multiplication. (Also called $S^1$ = the circle.)

Here - we can think of $G$ as $\mathbb{R}/2\pi\mathbb{Z}$

**ie)** $\left[0, 2\pi\right] / \{0 \sim 2\pi\}$ identify $0$ and $2\pi$ (equivariant)

Here $\mu$ has:

$$\int f \mu = \int_0^{2\pi} f(\theta) d\theta$$

so $\int 1 \mu = 1$ is why we divide by $2\pi$
An orthonormal basis of $L^2(U(1))$ is given by:

$$\psi_k(\theta) = e^{ik\theta}$$

**Defn. of inner product in $L^2$**

So has an inner product:

$$\langle \psi_k, \psi_l \rangle = \int_0^{2\pi} e^{ik\theta} e^{il\theta} \frac{d\theta}{2\pi} = \int_0^{2\pi} e^{i(l-k)\theta} \frac{d\theta}{2\pi} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

The hard part is showing $\{\psi_k\}$ form a basis.

Showing this uses the Stone-Weierstrass Thm.

So - given any $\phi \in L^2(U(1))$ we get

$$\phi = \sum_{k=-\infty}^{\infty} \hat{\phi}_k \psi_k$$

where

$$\hat{\phi}_k = \frac{1}{2\pi} \int_0^{2\pi} \psi_k(\theta) \phi(\theta) \frac{d\theta}{2\pi}$$

as usual whenever we have an o.n. basis called the Fourier transform.

The Fourier transform of $e^{ik\theta}$ gives $\hat{\phi}_k$ from $\phi$. 
The inverse Fourier transform is just:

\[ \phi(\theta) = \sum_{k=-\infty}^{\infty} \hat{\phi}_k e^{i k \theta} \]

This allows us to write any function in the guise of a sum of \( e^{i k \theta} \)'s, which we know how to differentiate, etc.

Question: Where are these functions \( e^{i k \theta} \) coming from?

2. What will we use in their place when we replace \( U(1) \) by some other group?

3. How do we do Fourier analysis on a compact group?

Answers:

- They came from the reps (the unitary ones) of \( G = U(1) \).

We can get a lot of 1-dim reps of \( U(1) \) as follows:

\[ \rho: U(1) \rightarrow \text{End}(\mathbb{C}) = \mathbb{C} \]

\[ \rho(e^{i \theta}) \]

We need: \( \rho(g g') = \rho(g) \rho(g') \)

\[ \rho(1) = 1 \]

\( \rho(e^{i \theta}) \) = \( \rho(e^{i \theta'}) \) = \( \rho(e^{i \theta}) \rho(e^{i \theta'}) \) and \( \rho(1) = 1 \).
We can use: raise to a power.

\[ \rho(e^{i\theta}) = e^{iK\theta}, \quad K \in \mathbb{Z} \]

Given 2 reps. \( \rho : G \rightarrow \text{End}(V) \) and \( \rho' : G \rightarrow \text{End}(V') \)

Then we get: \( \rho \circ \rho' : G \rightarrow \text{End}(V \oplus V') \) by

\[ (\rho \circ \rho')(g)(v, v') = (\rho(g)v, \rho'(g)v') \]

\( \text{End}(V \oplus V') \)

so need to feed in an elt of \( V \oplus V' \)

This is a rep. of \( G \). We say a rep. is irreducible if it's not of the form \( \rho \circ \rho' \) (unless \( \rho \) or \( \rho' \) is a rep. on a 0-dim'l vector space).

Thm: If \( G \) is a compact group, then every (finite-dim'l) rep. is a direct sum \( \rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_n \) of irreducible reps.

Thm: If \( G \) is abelian, all irreducible reps are 1-dim'l.

Note: our group \( U(1) = S^1 \) is both compact and abelian, so both above results apply.
Thm: Every irreducible rep. of $U(1)$ is (isomorphic to) one of the reps:

$$\rho(e^{i\theta}) = e^{ik\theta}, \quad k \in \mathbb{Z}$$

Note: $e^{ik\theta}$ come from all the irreducible reps. of our group $U(1)$.

A generalization of this idea will give us an orthonormal basis of $L^2(G)$ starting from all the irreducible reps. of $G$.

(look at matrix entries - give us the functs).

leading up to Peter-Weyl Thm.