If $A$ is an algebra there is a category whose objects are representations of $A$ and whose morphisms are intertwiners:

If $\rho: A \to \text{End}(V)$ are reps, an $\rho': A \to \text{End}(V')$ is a linear map $f: V \to V'$ s.t.

$$f\rho(a)v = \rho'(a)f v$$

Note: composition of intertwiners is an intertwiner.

If $A$ is a bialgebra this (category we mentioned above) is a monoidal category:

we can tensor reps $\rho: A \to \text{End}(V)$

$\rho': A \to \text{End}(V')$

to get

$$\rho \otimes \rho': A \to \text{End}(V \otimes V')$$

VIA:

Diagram:
If \( f : V_1 \to V_2 \), \( f' : V'_1 \to V'_2 \) are intertwiners, so is \( f \otimes f' : V_1 \otimes V'_2 \to V'_1 \otimes V_2 \), which we draw as above.

What extra structure must a bialgebra have for its category of reps to be a braided monoidal category?

We'll call such a bialgebra quasi-triangular (or braided).

Given 2 reps \( \rho : A \to \text{End}(V) \), \( \rho' : A \to \text{End}(V') \) of a bialgebra \( A \). Let's see what we need to define the braiding:

\[
B_{v,v'} : V \otimes V' \to V' \otimes V
\]

\( B_{v,v'} \) will be drawn as:

In the category of \( v \)-spaces, we had the braiding

\[
S_{v,v'} : V \otimes V' \to V' \otimes V
\]

Note - This is what we use for reps of a group.
Sometimes we can just use this for our $B_v,v'$ e.g.:

$$A = C[G] \ U_g$$

We secretly were doing this in the 1st quarter w/ $G = SU(2)$ or $SL(2,\mathbb{C})$.

We want to redraw our original braiding, $S_v,v'$ as:

This agrees w/ the fact that $\times \times = \times$ when we use $S_v,v'$. i.e) it doesn't matter which one is over or which one is under crossing.

Now, let's cook up an interesting $B_v,v'$ as follows (we want to define the new braiding in terms of the old one):

New braiding:

$$R \in A \otimes A$$

"an $R$-matrix"

Recall:

$$R \in A \otimes A = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

Recall a representation theorem...
Let's write our pictures in eqns:

\[ R = \sum R^{ij} e_i \otimes e_j \]

where \( e_i \) are any basis for \( A \), \( v \in V \), \( v' \in V' \).

\[ v \circ \otimes v' \]

\[ v \otimes R \otimes v' \]

\[ \sum R^{ij} v \otimes e_i \otimes e_j \otimes v' \]

\[ \sum R^{ij} \rho(e_i) v \otimes \rho'(e_j) v' \]

\[ S_{v,v'} \text{ (switch)} \]

\[ \sum R^{ij} \rho'(e_j) v' \otimes \rho(e_i) v \in V' \otimes V \]

Recall - the braiding last quarter had to satisfy certain properties, and so our braiding here will have to as well.
The defn. of a braided monoidal category requires:

1. 

\[
\begin{array}{ccc}
V 

& V \otimes V'' \\
\downarrow \\
B_{v,v''} & \\
\end{array}
\]

\[=\]

\[
\begin{array}{ccc}
V 

& V \\
\downarrow & \\
B_{v,v''} & \\
\end{array}
\]

2. 

\[
\begin{array}{ccc}
V \otimes V' & V'' \\
\downarrow & \\
V & V'' \\
\end{array}
\]

And 

\[
\begin{array}{ccc}
V \otimes V' & V'' \\
\downarrow & \\
V & V'' \\
\end{array}
\]

These are going to give us conditions \( R \) must satisfy.

Imposing 1 on 

\[
\begin{array}{ccc}
V 

& V' \\
\downarrow & \\
\downarrow & \\
\end{array}
\]

we get

\[
\begin{array}{ccc}
V 

& V' \\
\downarrow & \\
\downarrow & \\
\end{array}
\]

Recall, a representation always satisfies

\[
\begin{array}{ccc}
A 

& V \\
\downarrow & \\
\downarrow & \\
\end{array}
\]

\[=\]

\[
\begin{array}{ccc}
A 

& V \\
\downarrow & \\
\downarrow & \\
\end{array}
\]
So, RHS becomes

and LHS

Want to see conditions on $R$ to make this hold.
So—this will hold if

\[ R \Delta = R \rightarrow R \]

Imposing (2) — we similarly get —

\[ R \Delta = R \rightarrow R \rightarrow R \]

Note—3rd Reidemeister move follows from (1) 9, (2).

So—a bialgebra is **quasitriangular** if it is equipped with \( R \in A \otimes A \) s.t.

\[ \begin{align*}
R & \Delta = R \rightarrow R \rightarrow R \\
(1) & \\
(2) & \\
\text{Also—} & \\
\text{we want braiding to have an inverse.}
\end{align*} \]
And — in fact

\[ B_{v,v'} = \begin{array}{c} \circ \end{array} \text{ is invertible.} \]

(And perhaps more stuff needed to get a braided monoidal category.)

Henceforth — in this context, we write:

\[ B_{v,v'} = \begin{array}{c} \circ \end{array} \quad B_{v,v'}^{-1} = \begin{array}{c} \circ \end{array} \]

When is the category of representations of a quasitriangular bialgebra a symmetric monoidal category.

ie) when is

\[ B_{v,v'} = \begin{array}{c} \circ \end{array} = \begin{array}{c} \circ \end{array} ? \]

This will hold iff doing the braiding twice is the identity.

ie)
Using \( X = X \), this says

\[ \begin{array}{ccc}
V & \circlearrowleft & V' \\
\circlearrowleft & R & \circlearrowright \\
V' & \circlearrowleft & V \\
\circlearrowright & R & \circlearrowleft
\end{array} = \\
\begin{array}{ccc}
V & \circlearrowleft & V' \\
\circlearrowleft & R & \circlearrowright \\
V' & \circlearrowleft & V \\
\circlearrowright & R & \circlearrowleft
\end{array} \]

Do some thing to LHS as before = use fact that \( \rho, \rho' \) are reps.

Using fact \( \rho \) is a rep,

\[ S = \text{usual braiding, usual way of switching 2 vectors} \]

since \( \rho' \) is a rep.

this matters since our alg isn't nec commutative.
Fact about reps that we’re using:

\[ Y_{i'}^j = Y_{i'}^j \]

from prev pg.

So the braiding is a symmetry if

\[ R \quad = \quad i \quad i \]

In this case, we call our quasitriangular bialgebra triangular.

which quasitriangular bialgebra is lurking behind the Kauffman bracket skein relation:

\[ X = A + A^{-r} \cup \]
It's called: $U_q \mathfrak{sl}(2, \mathbb{C})$ - the "quantized" universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$.

(A is some function of $q$.)

First - Examples

1. If $G$ is a group, we've seen $C[G]$ is a bialgebra. In fact, it's triangular w/ $R = 1 \otimes 1 \in C[G] \otimes C[G]$.


2. If $g$ is a lie algebra, $U_q,g$ is triangular w/ $R = 1 \otimes 1$ 

$[x,y] = xy - yx$
If \( g = \text{sl}(2, \mathbb{C}) \), what's \( \text{U}_g \) like?

\( \text{sl}(2, \mathbb{C}) = \{ \text{2 \times 2 traceless } (\text{tr}(x) = 0) \text{ complex matrices} \} \)

4-dimensional since 4 entries, but \( \text{tr} = 0 \) condition knocks us down to 3-dimensional.

Our basis for \( \text{sl}(2, \mathbb{C}) \) is:

\[
E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}
\]

These satisfy

\[
[E, E] = [F, F] = [H, H] = 0 \quad \text{since } [x, y] = xy - yx
\]

**HW #4:**

\[
[H, E] = E, \quad [H, F] = -F, \quad [E, F] = 2H
\]

**Physics:**

- \( H \): "angular momentum along \( z \)-axis" (observable)
- \( E \): "raising operator"
- \( F \): "lowering operator" have the effect of raising or lowering ang. momentum.

If \( \psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \) then \( H \psi = \frac{i}{2} \psi \)

"spin-up" state

So \( \psi \) is a state where angular momentum along the \( z \)-axis is \( \frac{1}{2} \).

Electron—spin \( \frac{1}{2} \) particle
If \( \psi = (0) \in \mathbb{C}^2 \) then \( H \psi = -\frac{1}{2} \psi \).

\( \psi \) is "spin down".

Note: \( E \psi_\uparrow = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \psi_\uparrow \)

so that \( E \) is a raising operator. Note: \( E \psi_\downarrow = 0 \).

Similarly, \( F \) deserves the name lowering operator.

So— if \( g = SL(2, \mathbb{C}) \), \( U_g \) is the \underline{associative algebra} generated by \( E, F, \psi_\uparrow, \psi_\downarrow \) \underline{modulo the relations}

\[
EH - HE = E \\
HF - FH = -F \\
EF - FE = 2H
\]

and it's a \underline{bialgebra}, (recall— we can make any \underline{univ. enveloping alg} into a \underline{bialgebra} by defining \( \Delta \)).

with

\[
\Delta E = E \otimes 1 + 1 \otimes E \\
\Delta F = F \otimes 1 + 1 \otimes F \\
\Delta H = H \otimes 1 + 1 \otimes H
\]

and it's \underline{triangular} w/ \( R = 1 \otimes 1 \).
$U_q \mathfrak{sl}(2, \mathbb{C})$ is similar but it's an algebra generated by $E, F, K = q^h$, $K^{-1}$

$= q^h$ (where $q = e^h$ is a number)

We'll get back $U \mathfrak{sl}(2, \mathbb{C})$ in some sense as $h \to 0$ or $q \to 1$.

* $KE = qEK$

$KF = q^{-1} FK$

* $EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}}$

$U_q \mathfrak{sl}(2, \mathbb{C})$ becomes a bialgebra w/ $

\Delta E = E \otimes K + K^{-1} \otimes E$

* $\Delta F = F \otimes K^{-1} + K \otimes F$

$\Delta K = K \otimes K$

We'll check *
0. The $\hbar \to 0$ ($q \to 1$) limit:

$$K = e^{\hbar H} = \frac{1 + \hbar H + (\hbar H)^2 + \cdots}{2!}$$

first-order terms

So, $KE = q E K$ says

$$e^{\hbar H} E = e^{\hbar H} E e^{\hbar H}$$

expand out exponential

$$(1 + \hbar H + \cdots) E = (1 + \hbar + \cdots) E (1 + \hbar H + \cdots)$$

$$E + \hbar HE + \cdots = E + \hbar E + \hbar EH + \cdots$$

$\Rightarrow E = E \checkmark$ and

$$HE - EH = E \checkmark$$

(bring $\hbar EH$ to LHS)

(which we know is true!)

2. $EF - FE = \frac{e^{2\hbar H} - e^{-2\hbar H}}{e^\hbar - e^{-\hbar}}$

only 1st order terms

$$= \frac{4\hbar H}{2\hbar} = 2H \checkmark$$

(what we have in our relations)

3. $\Delta F = F \otimes K^{-1} + K \otimes F$

$$= F \otimes e^{-\hbar H} + e^{\hbar H} \otimes F$$

(lowest 0th order terms)

$= F \otimes 1 + 1 \otimes F$ (which is what we have)
Track 2: G - compact group

Some facts:

1. If \( \rho: G \rightarrow \text{End}(V) \) is a rep., then we can find an inner product on \( V \) st \( \rho \) is unitary, ie

\[
\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle \quad \forall \ v, w \in V, \quad g \in G
\]

2. If \( \rho: G \rightarrow \text{End}(V) \) is unitary & \( W \subseteq V \) is a subrep - ie \( \rho(g): W \rightarrow W \) \( \forall \ g \in G \).

Then \( W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall \ w \in W \} \) (all vectors \( \perp \) to \( W \)) is also a subrep.

Then \( V \) is a direct sum of the reps \( W, W^\perp \): \( V = W \oplus W^\perp \) as reps.

Note - we don't think of \( 0, 1 \) as irreducible just as we don't think of 1 as being prime.

3. We say \( V \) is irreducible if \( \{ 0, 1 \} \) and \( V \) are the only subreps of \( V \).

(Note - it's evil to say \( 0, 1 \) is irreducible)

Any rep \( V \) is a finite direct sum of irreducible reps \( V = \bigoplus_{i=1}^n W_i \).

(In fact, there is essentially a unique way to do this.)
Schur's lemma (Part I): Suppose
\[ \rho : G \rightarrow \text{End}(V) \text{ and } \rho' : G \rightarrow \text{End}(V') \] are
two irreducible reps of G.
Suppose \( f : V \rightarrow V' \) is an intertwiner:
\[ f \rho(g)v = \rho'(g)f(v) \quad \forall g \in G, \forall v \in V. \]

dichotomy: Then either:

(1) \( f \) is 1-1 \& onto (so it's invertible, so our
    reps are isomorphic)
or
(2) \( f = 0 \).

proof: Look at \( \ker f = \{ v \in V \mid f(v) = 0 \} \subseteq V \)

\[ \text{range } f = \{ v' \in V' \mid v' = f(v) \text{ for some } v \in V \} \subseteq V' \]

Claim: These are subreps of \( V, V' \) respectively.

But \( V, V' \) are irreducible, meaning
the only subreps of them are \( \{0\} \) or \( V \) (or \( V' \)).

\[ \Rightarrow \]

\( \ker f \) is either \( \{0\} \) or \( V \).
\[ \text{range } f \] is either \( \{0\} \) or \( V' \).
If $\ker f = V$, everything is sent to zero, so $f = 0$.

If $\text{range } f = \{0\}$, the image of everything is zero, so $f = 0$.

Otherwise—we have $\ker f = \{0\}$ and $\text{range } f = V'$.

So $f$ is 1-1 and onto.

Now we must check our claim:

**Proof of claim:**

1. $\ker f$ is a subrepresentation: i.e.
   
   Need: $v \in \ker f \Rightarrow \rho(g)v \in \ker f$
   
   or $f(v) = 0 \Rightarrow f\rho(g)v = 0$

   Now we use the fact that $f$ is an intertwiner.

   $\Rightarrow$ That is, $f\rho(g)v = \rho(g)f(v) = 0$ when $f(v) = 0$.

2. $\text{range } f$ is a subrep of $V'$.

   Need: $v' \in \text{range } f \Rightarrow \rho(g)v' \in \text{range } f$

   So $v' = f(v) \Rightarrow \rho(g)v' = f(w)$ for some $w \in V$

   for some $v \in V$

   But: $\rho(g)v' = \rho(g)f(v) = f\rho(g)v$

   $f$ is an intertwiner

   **End proof of**

   So we use $w = \rho(g)v$. **Claim a', lemma**
**Defn:** If \( \rho: G \to \text{End}(V) \) and \( \rho': G \to \text{End}(V') \)
are reps and \( f: V \to V' \) is an intertwiner
that's 1-1 and onto, we call it an equivalence
(or isomorphism) and we say the reps are equivalent.

**Moral:** Equivalent reps are "the same" for all practical purposes.

**Schur's Lemma (Part 2):** If \( \rho: G \to \text{End}(V) \)
is an irred. rep and \( f: V \to V \) is an intertwiner (then by part 1 \( f \) is either zero or 1-1 and onto) then in fact \( f = I \cdot v \) (a multiple of the identity) for some \( v \in V \).

**Proof:** Any \( f: V \to V \) can be written as

\[
\begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & \ddots & \ddots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & \lambda \\
\end{pmatrix}
\]

Jordan canonical form.

\[
T_i \to T_n
\]

in some basis.

we want:

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda \\
\end{pmatrix}
\]

\( T_i \) is a \( K_i \times K_i \) matrix

Incorrect!

here & below let

\[
T = \begin{pmatrix}
\lambda & 0 \\
0 & \lambda \\
\end{pmatrix}
\]

a \( K \times K \) matrix

then \((T - \lambda I)^k = 0 \quad (T - \lambda I) = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

\[(T - \lambda I)^2 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

raise to more powers, 1's move up.
If $f$ has blocks $T_1, \ldots, T_n$

the char. eqn! $$(f - \lambda_1 I)^{k_1} (f - \lambda_2 I)^{k_2} \ldots (f - \lambda_n I)^{k_n} = 0$$

Claim: Range $$(f - \lambda_1 I)^{k_1} \ldots (f - \lambda_i I)^{k_i} \ldots (f - \lambda_n I)^{k_n}$$

cross of one of them) is a subrep of $V$.  

($f$ is an intertwiner, so is $f^2$ since the comp. of intertwiners is again an intertwiner.

We saw in Schur (Part I) that the range of an intertwiner is a subrep: $f$ is an intertwiner so any poly in $f$ is also.

This range looks like: 

\[
\begin{cases}
\begin{pmatrix}
\ddots \\
\ddots \\
\ddots \\
\end{pmatrix} \\
\begin{pmatrix}
0 \\
\ddots \\
\ddots \\
\end{pmatrix} \\
\begin{pmatrix}
0 \\
\ddots \\
\ddots \\
\end{pmatrix} \\
\end{cases} \]

\[i^{th} \text{ block}\]

this is supposed to be a sub-rep of $V$, but $V$ is irreducible!

So, since $V$ is irreducible, this subrep must be either $\emptyset$ or $V$.

The only way this
proof:

If $\text{alg. closed}$, so $f$ has an eigenvalue, $\lambda$. Consider $(f-\lambda I)$.

This has nontrivial kernel: $\ker (f-\lambda I) \neq 0$.

$\Rightarrow \ker (f-\lambda I) = V$ since $V$ is irreducible.

(only subrep are $\{0\}$ and $V$ and $\neq 0$).

Thus $f = \lambda I$. $\square$