5/22/01

Redo 1st quarter's work on spin networks replacing

\[ X = \underset{V}{\bigcup} + \underset{U}{\bigcup} \]

where lines are labelled by \( V = \mathbb{C}^2 \)
and \( U \) is \( w : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C} \) the symplectic structure

\[ w(e_1 \otimes e_2) = 1 \quad w(e_1 \otimes e_1) = 0 \]
\[ w(e_2 \otimes e_1) = -1 \quad w(e_2 \otimes e_2) = 0 \]

which determines \( s^+ \)

\[ s^+ \quad \bigcup \bigcup = / = \bigcup \]

replace \( w / \)

\[ X = A \bigcup + A^{-1} \bigcup \quad 0 \neq A \in \mathbb{C} \]

Also -

\[ X' = A \bigcup + A^{-1} \bigcup \]
\[ \sigma = d = -(A^2 + A^{-2}) \]

In the 1st quarter we defined "symmetrizers"

\[ p_s = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \]

perm group

This \( p_s : \mathbb{V}^n \to \mathbb{V}^n \) has \( p_s^2 = p_s \) so we let \( j = S^n \mathbb{V} = \text{Range } p_s \)

where \( j \) is a number, related to \( n \).

\[ \begin{array}{c}
2j = n \\
j = 0, \frac{1}{2}, 1, \ldots \\
n = 0, 1, 2, \ldots
\end{array} \]

dim space = \( 2j + 1 \)

Ex)

\[ j = \frac{1}{2} \]

1 strand \( n \), so \( j = \frac{1}{2} \)

Everything is built from \( \mathbb{V} = \mathbb{C}^2 \) and symplectic structure \( \omega \) on it.

The linear transit that preserve \( \omega \) are precisely those \( \psi \) with \( \det = 1 \).
We saw

\[ \omega(gv \otimes gw) = \omega(v,w) \quad \forall \; v,w \in V\]

iff \( g : \mathbb{C}^2 \to \mathbb{C}^2 \) (linear) has determinant 1.

ie) \( g \in SL(2, \mathbb{C}) \).

So \( SL(2, \mathbb{C}) \) acts as symmetries.

(rep, on groups, we can then tensor them.) \( SL(2, \mathbb{C}) \) has a representation on \( V = \mathbb{C}^2 \) and \( V^{\otimes n} \) gets a rep of \( SL(2, \mathbb{C}) \) since we can tensor reps.

And - \( \sigma : V^{\otimes n} \to V^{\otimes n} \) are intertwining operators, or “intertwiners,” for all \( \sigma \in V^{\otimes n} \) so that

\[ p_s = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \] is an intertwiner!

So, \( j = \text{Range } p_s \subseteq V^{\otimes n} \) is a subrepresentation.

Here we’re being sloppy. We’re writing

iff \( gY = p_s g \phi \) \( \forall \phi \in V^{\otimes n} \)

since \( p_s \) is an intertwiner.

We used to say \( p(g)Y \) now we drop the name \( p \).

gY instead of \( p(g)Y \).
Recall: $f: V \rightarrow W$ if $f$ is an intertwining:

$p \sim p'$ if $f \circ p = p \circ f$.

Again:

$\psi = p_\phi$

iff

$g \psi = g \circ p_\phi$

iff

$g \psi = p_\phi \circ g$

iff

$g \psi \in \text{range } p_\phi$

When $V, W$ reps of groups $V \otimes W$, this map $V \otimes W$

$\times$

$W \otimes V$

is an intertwining.

We call $j \leq V^\otimes$ the "spin-$j$ representation" of $SL(2, \mathbb{C})$. And in particular, $\frac{1}{2} = V$ is called the "spin-$\frac{1}{2}$ representation" of $SL(2, \mathbb{C})$.

Thm: Every irreducible (finite dim'l) representation of $SL(2, \mathbb{C})$ is equivalent to one of these spin-$j$ reps where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$

and $\dim(j) = 2j + 1$.

Now let's make up "$q$-deformed" symmetrizers analogous to $p_\psi$, but for Kauffman bracket relations.
A tangle with \( n \) incoming strands and \( n \) outgoing strands gives us an operator from \( V^\otimes n \) to \( V^\otimes n \).

Let \( A, U, X, X \) be operators such that
\[
X = A X + A^{-1} U
\]
and
\[
0 = - (A^2 + A^{-2})
\]
Any tangle with \( n \) strands in, \( n \) strands out gives an operator \( T: V^\otimes n \to V^\otimes n \).

Linear combinations of these operators form not only a vector space, but also an algebra where multiplication is defined:

The product is:

\[
\begin{align*}
T & \quad S \\
\text{and} & \\
T \otimes S & = S \otimes T
\end{align*}
\]
This algebra formed by linear combinations of our linear operators is called the Temperley-Lieb alg, $T_n$.

Now, we want to find "q-deformed symmetrizers" $p \in T_n$.

$T_n$ is generated as an algebra by $e_1, \ldots, e_{n-1}$.

$e_1 = \begin{array}{c}
\cap \\
\cup \\
\vdots
\end{array}$

cap, cup takes up 2 strands.

$e_2 = \begin{array}{c}
\cap \\
\cap \\
\vdots
\end{array}$

$e_3 = \begin{array}{c}
\cup \\
\cup \\
\vdots
\end{array}$

$e_{n-1} = \begin{array}{c}
\cup \\
\cup \\
\vdots
\end{array}$

Any elt of Temperley-Lieb alg is a linear comb. of products of $e_1, \ldots, e_{n-1}$.

ie) $T_n$ consists of lin. comps. of products of these $e_i$.
so in TL$_3$. Want to express this as a
lin comb of e$_i$.
We can use skein relation.

by Kauffman bracket skein rel:
\[ X = A \Upsilon - A^{-1} \Upsilon e_1 \]
\[ \Upsilon \rightarrow \Upsilon = Ae_2 + A^{-1} e_2 e_1 \]

comes 1st at top

(above horiz line is e$_1$), below horiz line is e$_2$.

Recall:

Kauffman bracket skein rel:
\[ X = A \Upsilon + A^{-1} \Upsilon \]

Also can use:
\[ X = A^{-1} \Upsilon + A \Upsilon \]
\[ \Upsilon = -(A^3 + A^{-2}) \]
These $e_i$ satisfy some relations:

$$e_i e_j = e_j e_i \quad \text{if} \quad |i-j| = 2$$

They commute since not tangled up in one another. We can stretch things to make these pictures look like each other.

But:

$$e_2 \neq e_3$$

So, we don't have $e_i e_j = e_j e_i$ if $|i-j| = 1$.

Instead, we have $e_i e_{i+1} e_i = e_i$.
And:

(2) \( e_{i+1} e_i e_{i+1} = e_{i+1} \)

And finally:

(3) \( e_i^2 = d e_i \)

Recall \( d = 0 \)

\[ d = -(A^2 + A^{-2}) \]
Thm: \( T L_n \) is the algebra freely generated by \( e_i \) modulo precisely these relations:

\[
e_i e_j = e_j e_i \quad \text{if } |i-j| \geq 2
\]

\[
e_i e_i + e_i = e_i, \quad e_i e_i e_i = e_i, \quad e_i^2 = d e_i
\]

In the first quarter, we saw \( p_s : V^\otimes n \to V^\otimes n \) satisfied:

\[
\text{symmetrizer } q, \text{ stick in a cap, cup},
\]

\[
\text{and } \frac{1}{2} \, ( \frac{1}{n} + \frac{1}{n} ) = 0
\]

(having cup above)

Recall:

\[
\text{Ex): } \quad \frac{1}{2} \left( \frac{1}{n} \right) = \frac{1}{n} + \frac{1}{n}
\]

\[
\text{Ex): } \quad \text{but } U = - \frac{1}{n} \text{ so that }
\]

\[
U = \frac{1}{n} + \frac{1}{n} = 0.
\]

\[
\text{get } x = \beta = 1. \text{ since } \alpha^2 = \beta \alpha \text{ is}
\]
Let's generalize this to the new context:

If \( A \) is not a root of unity: ie) \( A^k \neq 1 \ \forall \ k=1,2,\ldots \)

**Thm:** For all \( n \), \( \exists \) a unique nonzero \( p \in TL_n \) st

so \( p \) is linear

1. \( p^2 = p \) (projection operator)
2. \( p_{e_i} = 0 \ \forall \ i \leq n-1 \)
3. \( e_i p = 0 \ \forall i \leq n-1 \)

We call \( p \) the "\( q \)-deformed symmetrizer" or "\( q \)-symmetrizer" and denote it by

\[ p \]

**Proof:** First we'll prove uniqueness then existence.

Suppose \( p \) and \( q \) both satisfy these properties. Want to show \( p = q \).

Write \( p = \alpha 1 + u \), \( \alpha \in \mathbb{C} \)

where \( u \) is a finite linear comb. of nontrivial products of \( e_i \)'s, since \( TL_n \) is generated by \( e_i \)'s.
Similarly, write
\[ q = \beta 1 + v \]

Then
\[ pq = p(\beta 1 + v) = \beta p \]

\[ pv = 0 \text{ since } v \text{ is a lin. comb of products of } e_i \]
and \[ pe_i = 0 \ \forall i \].

But
\[ pq = (\alpha 1 + u)q = \alpha q \]

since \( e_i q = 0 \ \forall i \)

so \( u q = 0 \) since \( u \) is a comb of products of \( e_i \).

Thus, \( \beta p = \alpha q \).

Now, use the fact that \( p \) and \( q \) are projections.
So,
\[ \beta^2 p^2 = \alpha^2 q^2 \]

\[ p^2 = p, \ q^2 = q \text{ by property 0} \]

\[ \Rightarrow \beta^2 p = \alpha^2 q \]

\[ \beta(\alpha q) = \alpha^2 q \text{ and } q \neq 0 \Rightarrow \beta \alpha = \alpha^2 \]

\[ \Rightarrow \alpha = \beta \]

But \( \alpha = \beta = 1 \) since \( p^2 = p \).

(i.e.)
\[ pp = p(\alpha 1 + u) = p \text{ but } pu = 0 \text{ since } pe_i = 0 \]

\[ \Rightarrow \alpha p = p \]

\[ \Rightarrow \alpha = 1 \text{ (since } p \neq 0) \text{ Similarly } \beta = 1. \]
Thus, \( pq = p \) and \( pq = q \)

so \( p = q \). \( \checkmark \)

Existence: For \( n = 1 \)

\[ + = \]

does the job. i.e. \( 1 \in TL_1 \) is our \( p \) for \( n = 1 \).

We define

\[ + \]

for higher \( n \) recursively:

\[ + \]

where

\[ \Delta_n = \]

ex)

\[ \Delta_1 = \bigcirc = \bigcirc = \Delta = - (A^2 + A^{-2}) \]
Let's show by induction that it's a projection operator. 
(i.e. \( p^2 = p \))

\[
\begin{align*}
(n+1) & = n + 1 \\
n & = 3 \\
(n-1) & = 2
\end{align*}
\]

\[
(a - b)^2 = a^2 - b + ab + b^2 \quad a, b \text{ not commut.}
\]

\[
p^2 = \frac{\Delta_{n-1}}{\Delta_n} + \left( \frac{\Delta_{n-1}}{\Delta_n} \right)^2
\]

Now we can use the inductive hypothesis:

\[
\begin{align*}
\hat{a} & = \hat{a} \\
& = \frac{\Delta_{n-1}}{\Delta_n} + \left( \frac{\Delta_{n-1}}{\Delta_n} \right)^2
\end{align*}
\]

We need to get rid of the 2 and circle.

**Lemma:**

\[
\frac{\Delta_{n-1}}{\Delta_{n-1}} = \frac{\Delta_n}{\Delta_{n-1}}
\]
Using the lemma, we get:

\[ p^2 = \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots \\
    n-1
\end{array} = \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{array} - \frac{2 \Delta_{n-1}}{\Delta_n} \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{array} + \frac{\Delta_{n-1}}{\Delta_n} \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{array} \\
\]

= by defn

So we need only to prove the lemma.

proof of lemma: Hard part is to show

\[ \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{array} = x \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{array} \]

As in the proof of uniqueness,

\[ p = \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{array} = 1 + u \] where \( u \) is a lin. comb. of nontrivial products of \( e_i \)'s.

\[ q = \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{array} = x \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{array} + v \] where \( x \in \mathbb{C} \) and \( v \) is a lin. comb. of nontrivial products of \( e_i \)'s.
\[ q_p = (\alpha + v) p \]

\[ = \alpha p \quad \text{since } v_p = 0 \]
\[ \quad \text{since } e^i p = 0 \forall i \]

but also:
\[ q_p = q(1 + u) = q \]

since
\[ gu = \begin{array}{c}
\hline
u \\
\hline
0
\end{array} = \begin{array}{c}
\hline
u \\
\hline
0
\end{array} = 0 \]

by the inductive hyp.

\[ u \text{ has } e_i \text{'s in it} \]

so we have \( \alpha p = q \), which is what we were trying to show.

Need to find \( \alpha \):
\[ \begin{array}{c}
\hline
0 \\
\hline
\end{array} = \alpha \begin{array}{c}
\hline
0 \\
\hline
\end{array} \]

\[ \Rightarrow \]
\[ \begin{array}{c}
\Rightarrow \\
\hline
0 \\
\hline
\end{array} = \alpha \begin{array}{c}
\hline
0 \\
\hline
\end{array} \]

\[ \Delta_n = \alpha \Delta_{n-1} \quad \text{so, } \alpha = \frac{\Delta_n}{\Delta_{n-1}}. \]

end lemma
Now - why is
\[ \times e_i \] (in either order)
equal to zero?

Recall:
\[ n+1 \]
\[ \frac{\Delta_{n-1}}{\Delta_n} \]

**Inductive Hypothesis:** Assume true for \( n \), prove true for \( n-1 \).

If \( 1 \leq i \leq n-3 \)

\[ 0 \] by hyp. \[ 0 \] what's in box covered by inductive hyp

\( e_i p = 0 \)
(and similarly \( pe_i = 0 \))

**Note:** recall sticking side by side - tesorning
\[ \begin{array}{c}
F
\end{array} \begin{array}{c}
v
\end{array} \begin{array}{c}
u'
\end{array} = \begin{array}{c}
F \otimes v
\end{array} \begin{array}{c}
v'
\end{array} \begin{array}{c}
w
\end{array} \begin{array}{c}
w'
\end{array} \]
and if \( F \circ G \) are zero, then \( \otimes \) is too.

If \( F \circ G = 0 \), then \( F \circ F \circ G = 0 \) and \( F \otimes G = 0 \).

*All our pictures represent multiplication (composition or tensoring) so if a part is zero, all is zero.*
Now, if \( i = n-2 \)

\[ e_{ip} = 0 \]

\[
\text{same result.}
\]

If \( i = n-1 \)

\[
\text{But our lemma says:}
\]

\[ \|O\| = \frac{\Delta_n}{\Delta_{n-1}} \]

so we get (from above)

\[
\|U\| = \|U\| - \|U\| = 0
\]

so \( e_{ip} = 0 \) and similarly \( p_{ei} = 0 \).

Note: we haven't used hypothesis that \( A \) isn't a root of unity.

Claim: If \( A \) isn't a root of unity, then

\[ \Delta_n \] is never zero so this argument is fine.

(We divide by \( \Delta_n \) everywhere).
continued claim:

If \( A \) is a root of unity, for some \( n \), \( \Delta_n \) will be zero so we cannot continue this recursion.

Examples:

1. \( n = 1 \)

\[
\Delta_1 = \bigcirc = \bigcirc = d = -(A^2 + A^{-2})
\]

So, \( \Delta_1 = 0 \) if \( A^2 = -A^{-2} \)

ie. \( A^4 = -1 \)

ie. \( A^8 = 1 \) (an eighth root of unity)

2. \( n = 2 \)

\[
\Delta_2 = \bigcirc = \frac{\Delta_0}{\Delta_1} \bigcirc
\]

Recall \( \Delta_0 = 1 \) (empty picture is one)

\[
\Delta_2 = \bigcirc = \frac{1}{d} \bigcirc
\]

Let's check that this is a projection operator.
\[ \# = \| \| - \frac{2}{d} \bigcup + \frac{1}{d^2} \bigcup \]

\[ = \| \| - \frac{2}{d} \bigcap + \frac{1}{d} \bigcap \]

\[ = \| \| - \frac{1}{d} \bigcap \]

So, \( \Delta_2 = \bigcirc = \bigcirc - \frac{1}{d} \bigcirc \)

\[= d^2 - \frac{1}{d} (d) \]

\[= d^2 - 1 \]

\[= (A^2 + A^{-2})^2 - 1 \]

\[= A^4 + 1 + A^{-4} \]

So: \( \Delta_0 = 1 \)

\( \Delta_1 = -(A^2 + A^{-2}) \)

\( \Delta_2 = A^4 + 1 + A^{-4} \)
HW: Use the recursive defn. of $\Delta_n \equiv \frac{\Delta^{n+1}}{\Delta^{n-1}}$ to get a recursion relation for

$$\Delta_n = \text{(close up projection operator)}$$

Use this to get a closed form formula for $\Delta_n$.

Guess: $\Delta_3 = -(A^6 + A^4 + A^2 + A^{-2})$ ??

When $A = 1$, we already know $\Delta_n = (n+1)(-1)^n$

When $A \neq 1$, we say

$$\Delta_n = [n+1]_q (-1)^n$$ where $[n+1]_q$ is called a "q-integer"