4/30/02  Connections on principal bundles

(book talks about connections on vector bundles)

* All forces of nature are connections
  (ex- vector potential)

All forces of nature are described by connections:

* Maxwell's eqns  (connection = vector potential, really 1-form $A$ on spacetime
  \[ F = dA \]
  \[ + *dF = j \]
  (A is a connection)

* Yang-Mills eqns  (describe weak, strong nuclear forces)

Here - a connection is roughly a 1-form w/ values in some Lie algebra.
For EM, the Lie alg. was $IR = U(1)$
  (Lie alg of $U(1)$)

* $SU(2)$ gives the weak force
  $SU(3)$ gives the strong force

  \[ F = dA + \frac{1}{2} [A, A] \]

  \[ + *dF + *[A, +F] = J \]

* Einstein's eqns  (describes gravity)

  use Lie alg $so(3,1)$. 
We'll focus on principal $G$-bundles and define connections on those, the Lie algebras associated to the different forces of physics are really the Lie algebras of different choices of $G$ "be gauge group."

Recall: For any Čech 1-cocycle $g_{ij}$ on our "base" manifold $B$:

$$g_{ij} : U_i \cap U_j \to G$$

we get $G$-bundles, in particular, a principal $G$-bundle, $P_B$, i.e. a bundle where fiber is $G$ and we think of $G \subset \text{Diff}(G)$ via left translations:

$$g \mapsto L_g$$

$$L_g \in \text{Diff}(G)$$

$$L_g(x) = gx \quad \forall x \in G.$$

Example: Let $G = \mathbb{R}$

$$\rho^{-1}(b) \quad \text{(diffeo to } \mathbb{R})$$

The fibers all look like $\mathbb{R}$ but aren't isomorphic in any "god-given" way.

The point: * * The fibers $\rho^{-1}(b)$ ($b \in B$) are diffeo to $\mathbb{R}$, but not in any god-given best way.
We know \( 0 \in \mathbb{R} \), but we can't tell on the fiber \( p^{-1}(b) \) where 0 is.

We get a diffeomorphism \( p^{-1}(b) \xrightarrow{\sim} \mathbb{R} \) if we pick a trivialization of \( p \).

In fact, it suffices to pick a local trivialization over \( U \subseteq \mathcal{B} \) where \( U \) is an open set and \( b \in U \).

\[
\begin{array}{c}
\text{Picking a trivialization means a commutative diagram, w/ trivial IR bundle.}
\end{array}
\]

\[
\begin{array}{c}
\text{the trivialization \( t \) is a diffeo, so we can look at inverse image of "zero" points over in \( P/U \). So, we call "zero" in \( P/U \) something that gets sent to \((b,0)\) via \( t \).}
\end{array}
\]
Example: let $B = \text{our universe (some 4-manifold)}$

$TB = \text{tangent bundle}$

so fibers, are like

$F = \mathbb{R}^4$

We have no diffeo bet. $F^4$, $\mathbb{R}^4$ in a "best" way.

$FB = \text{frame bundle - a principal } \text{GL}(4)\text{-bundle}$

(4x4 invert. matrices)

$FB$

$\downarrow$

$p$

$\downarrow$

$B$

$p^{-1}(b) = \{\text{all bases of the tangent space } T_b B \text{ at } b\}$

(recall - for a principal bundle, the fibers look like the group).

Here - the fibers $F_b B = p^{-1}(b)$ are all diffeomorphic to $G = \text{GL}(4)$, since if we pick a basis of $T_b B$, every other basis can be written out as a matrix w/r/t the chosen one, so we get

$F_b B \overset{\sim}{\longrightarrow} \text{GL}(4)$

But as before, this diffeo

$F_b B \overset{\sim}{\longrightarrow} \text{GL}(4)$

is not god-given (we made a choice).
Theorem: If $B$ is any $n$-dimensional manifold, its frame bundle $\pi^*F_B \to B$ is a principal $GL(n)$ bundle.

To answer the question, we want to pick up our basis, bring over to $b'$, and then answer question. But this requires carrying our basis over without rotating it, which we can't do since spacetime is curved. This is the definition of curvature.

Ex) but go down side, then right don't agree
Given a frame \( e \in F_b B \) and a point \( b' \in B \), how can we "carry" \( e \) over to \( b' \)? The answer could depend on the path from \( b \) to \( b' \).

It also depends on the "connection" — a geometrical structure on the frame bundle \( F_b B \).

More generally — we can define a connection on a principal \( G \)-bundle?

\[ \begin{array}{ccc}
P & \xrightarrow{p} & B \\
\downarrow & & \\
B & & \\
\end{array} \]

Ex) here, \( G = \mathbb{R} \)

\[ \begin{array}{ccc}
P & \text{e} & \text{e'} \\
\downarrow p \\
B \end{array} \]

\( e \in p^{-1}(b) \)

Want to take \( e \) to \( e' \). So, choose a path from \( b \) to \( b' \).

If this were a trivial bundle, we could carry \( e \) horizontally across.

We'd like to carry \( e \in p^{-1}(b) \) to \( e' \in p^{-1}(b') \) without twisting it around. A connection allows us to do this.
We want to carry e to e' st the curve e(t) in P is "horizontal" - but we don't know what this means if we don't have a god-given way of comparing points in any fiber \( p^{-1}(b) \) w/ points in \( E \).

A connection will be a choice of "horizontal."

![Diagram of tangent space](TeP)

We want to pick a "horizontal subspace"

\[ H_e \subseteq T_e P \]

A tangent vector in here will be declared to be horizontal.

If we choose a tang. vector \( \uparrow \) to be "horizontal", we see that \( dp \) sends this tang. vector to zero, so it wasn't "horizontal".

(differenti'ae) sends tang. vectors to tang. vectors

\[ P \quad \uparrow \quad e' \quad \text{Not okay!} \]

\[ \downarrow \quad P \quad e \]

\[ B \]
We'll figure out some conditions that the spaces $H_e$ need to satisfy.

1. A vector $0 \neq v \in H_e$ should not be vertical; we say $v \in T_e P$ is vertical if $dp(v) = 0$.

\[
\begin{align*}
T_e P & \quad \text{p}(e) = b.
\end{align*}
\]

\[
T_b B
\]

Note: We can define $V_e \subseteq T_e P$ the vertical subspace without any choice of extra structure; just define

\[
V_e = \left\{ v \in T_e P \mid dp(v) = 0 \right\}
\]

= ker $dp$

But choosing $H_e$ really involves a choice.

In short, we want $H_e \cap V_e = \{0\}$, and

We can decompose a tang vector into a hori of vert. part.

$T_e P = H_e \oplus V_e$ (meaning every elt of $T_e P$ is a pair of something hori, something vert)
Since we can decompose any tangent vector in $T_e P$ into a horizontal, vertical part, we can look at tangent vectors $d\pi_i$ want vertical part to be zero. (This gets no horizontal vectors.)

If $T_e P \cong T_e \mathbb{E} \oplus T_e \mathbb{E}$ then consider:

$$dp : T_e P \to T_b B$$

$dp$ kills everything in $T_e \mathbb{E}$, but is 1-1, onto on $T_e \mathbb{E}$.

$T_e \mathbb{E} = \ker dp$.

(3) $dp$ annihilates $T_e \mathbb{E}$, but restricts to an iso on $T_e \mathbb{E}$.

$$dp : T_e \mathbb{E} \sim \to T_b B$$

$dp$ is 1-1 on $T_e \mathbb{E}$: Since if $v \in T_e \mathbb{E}$ had $dp(v) = 0$, then it would have to be vertical, so $v \in T_e \mathbb{E} \cap T_e \mathbb{E} = \{0\} \implies v = 0$.

$dp$ maps $T_e \mathbb{E}$ onto $T_b B$:

**Pick a local trivialization:**

$$P \leftarrow u \to U \times G$$

$p'$ is onto, and $p \cong p'$.
dp maps $T_e P$ onto $T_b U$ since $dp'$ maps $T_e (U \times G)$ onto $T_b (U)$.

$$dp' (v/w) = v, \ v \in T_b (U)$$

and bundles $P/\mathcal{U}$ is iso to $U \times G$

\[ \begin{array}{c}
U \\
\downarrow \\
\mathcal{U}
\end{array} \]

2) We need $He$ to vary smoothly with $e$ (depends smoothly on $e \in P$).

3) $P$

\[ \begin{array}{c}
\mathcal{H}e \\
\downarrow \\
\mathcal{P}
\end{array} \]

$G = \mathbb{R}$

It would be nice if once we said is horizontal, then on the same fiber, we get all things horiz. to look the same.

ie) If $e, e'$ are in the same fiber, we want $He$ and $He'$ to be "parallel"
Recall: \( G \) has a right action on \( P \), preserving each fiber.
Moreover, given any \( e, e' \) in the same fiber, \( \exists! g \in G \) s.t.
\[
   eg = e' \quad (\text{right action})
\]
This is because each fiber is "like" (iso, but not in a god-given way) \( G \) w/ action like right mult.

Then - \( g \) acts on \( P \), giving a map \( f \)
\[
f: P \rightarrow P \quad \text{and thus} \quad f(e) = e'
\]
\[
df: T_e P \rightarrow T_{e'} P
\]
and we require that
\[
df: He \rightarrow He'.
\]

**Definition:** A connection on our principal \( G \)-bundle \( P \) over \( M \) is a choice of "horizontal" subspace
\[
   He \leq T_e P \quad \forall e \in P \quad \text{s.t.}
\]
\[
   \begin{align*}
   &1 \quad He + Ve = T_e P \\
   &2 \quad He \text{ varies smoothly w/ } e \\
   &3 \quad \text{If } e' = f(e) \text{ where } f: P \rightarrow P \text{ comes from right action of } g \in G \text{ on } P, \text{ then } df(He) = He'.
   \end{align*}
\]