5/16/02

\[ C \text{-} \text{Cat} = \text{the category whose objects are category objects in } C \text{ and whose morphisms are functors (functor-morphisms) between these.} \]

<table>
<thead>
<tr>
<th>C</th>
<th>C-Cat</th>
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<tbody>
<tr>
<td>\text{Set}</td>
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<td>\text{Top}</td>
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<td>\text{Diff}</td>
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<td>\text{Grp}</td>
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<tr>
<td>\text{Ab Grp}</td>
<td>\text{Ab Grp} - \text{Cat}</td>
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<tr>
<td>(2-term chain complex of abel. groups)</td>
<td>\text{ Vect} - \text{Cat}</td>
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<tr>
<td>(2-term chain complexes of v. spaces ( X_0 \rightarrow X_1 ))</td>
<td>[\text{Diff} - \text{Grp}] - \text{Cat}</td>
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<td>(strict) Lie 2-gps</td>
<td>\text{Cat} - \text{Cat} = \text{double category}</td>
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\text{Lie Grp} is a group object in \text{Diff Grp}.
space of objects, space of morphisms

"topological categories"

(people usually assume the space of objects is discrete)

problem! Diff doesn't always have pullbacks

$X \in C$ - cat has an object of objects, $O = X_0$

object of morphisms, $M = X_1$

composition: need source of target to match

(target of 1st = source of 2nd)

\[ X_1 \xrightarrow{X_0} X_1 \xrightarrow{X_1} X_1 \]

Ex)

$\{(x, y) \mid xy = c\}$

$\mathbb{R}^2$

$(x, y)$

$\mathbb{R}$

$xy$

Want pairs of something in $\mathbb{R}^2$, something in $\{\ast\}$

If $xy = c$ parabola

but at $(0,0)$, deriv of map $(x, y) \mapsto xy$

is zero.
- Pullback exists in Diff only if $c \neq 0$.
- Transversality - condition that guarantees pullbacks.

* In the defn. of category object - we demand that the necessary pullbacks exist.

3. called a "groupal category" = 2-group (strict) recall - we talked about a categorical group (a group object in $\text{Cat}$)

4. $X_0 \leftarrow d^\alpha X_1$
   $X_1$, $X_0$ abelian groups
   (don't need $d^2 = 0$ since only 1 map)

5. categorify Gauge Theory - categorified Vector Spaces

We've mentioned 2 concepts:

1. "group objects in $\text{Cat}$" = "categorical groups"
2. "category objects in $\text{Grp}$" = "groupal categories"

These are the same!

**Thm:** $\text{Grp-Cat} \cong \text{Cat-Grp}$

\[
\uparrow \text{equivalence of categories}
\]
proof: This is an example of "commutativity of abstraction." Both "gap object" and "category object" in any category w/ finite limits.

More generally—suppose $X$ and $Y$ are structures that we can define using commut. diagrams w/ finite limits. Then $C - X$ (an $X$-object in $C$) and $C - Y$ (a $Y$-object in $C$) both exist and have finite limits when $C$ does.

Then we can talk about a $Y$-object in $(C - X)$ and $X$-object in $(C - Y)$. So

$$(C - X) - Y \cong (C - Y) - X \quad (*)$$

Here: $C - X$ is the category of $X$-objects in $C$, etc.

In fact, there is a category w/ finite limits $X$, called "the walking $X" or "the Platonic idea of $X."$

Suppose $X$ is the concept of group. Then the walking group, ie) $X$ is a category w/ finite limits containing:

1) an object $G$ (idea of a gap)
2) a morphism $m: G \times G \to G$
3) a morphism $i: 1 \to G \quad (1 \text{ is terminal obj})$
4) a morphism $\text{inv}: G \to G$
satisfying exactly the relations in defn of group:

1) left/right unit laws
2) associative law
3) left/right inverse laws

and their consequences.

(This is similar to defining something via generators & relations.)

For more precision—see the concepts of sketch and finite limits theory in Barr & Wells' *Toposes, Triples & Theories*.

$X$ is also called the "theory of groups."

(Theory of a group)

* A group object in a category $C$ w/ finite limits turns out to be precisely a functor

$$ F: X \longrightarrow C $$

which preserves finite limits, or is left exact.
A homomorphism between 2 group objects: \( F, F' : X \to C \)
is any natural transformation,
\( \alpha : F \Rightarrow F' \)

So \( C - X \) (\( X \) objects in \( C \)) is the category called:
\[ C - X = \text{Lex}(X, C) \quad \text{(left exact)} \]

where objects in \( \text{Lex}(X, C) \) are left exact functors \( F : X \to C \) and morphisms are nat. transf. between them.

We wanted to show:
\[ (C - X) - Y \cong (C - Y) - X \]

so we need:
\[ \text{Lex}(Y, \text{Lex}(X, C)) \cong \text{Lex}(X, \text{Lex}(Y, C)) \]

(Seen before when \( \text{Lex} \) was hom. \( X, Y, C \) were \( R \)-modules)

Given categories \( C \) \& \( D \) w/ finite limits, there is a
category w/ finite limits called \( C \otimes D \) s\( t\)
\[ \text{Lex}(C \otimes D, E) \cong \text{Lex}(C, \text{lex}(D, E)) \]

\( E \) another cat w/
finite limits

But \( C \otimes D \cong D \otimes C \), so —
Here - $C \otimes D$ starts w/ $C \times D$ & throw in all finite limits.

Since $C \otimes D = D \otimes C$, we get

$$\text{lex}(\bar{Y}, \text{lex}(\bar{X}, C)) \cong \text{lex}(\bar{Y} \otimes \bar{X}, C)$$

$$\cong \text{lex}(\bar{X} \otimes \bar{Y}, C)$$

$$\cong \text{lex}(\bar{X}, \text{lex}(\bar{Y}, C))$$

In fact:

**Thm.** TFAE

1. Category objects in Grp
2. Group objects in Cat
3. strict 2-gaps (from last quarter)
   \text{ie- strict 2-cats w/ one object and all morphisms and 2-morphisms invertible}
4. crossed modules.

We've seen 1, 2 are equivalent.

What about 1, 2, 3?
Let $C$ be a category object in $\text{Grp}$. It has a group of objects $C_0$. But we can compose group objects, so we'll draw them as arrows:

\[ g \in C_0 \]

\[ g \xrightarrow{h} \quad \text{gives } gh \in C_0. \]

It also has a group of morphisms, $C_1$ and source $s$ target homomorphisms:

\[ s, t : C_1 \to C_0 \]

We draw $h'g \to g'$ as $(h \in C_1), \ g, g' \in C_0$

\[ \bullet \quad \downarrow h \quad \bullet \]

\[ g = s(h) \quad g' = t(h) \]

Given $h, h' \in C_1$, we draw their product in $C_1$ as

\[ \bullet \quad \downarrow h \quad \bullet \]

\[ \bullet \quad \downarrow h' \quad \bullet \]

since $s, t$ should be homos.

*This is mult. in the group $C_1$.

\[ s(hh') = s(h)s(h') \]

\[ t(hh') = t(h) + t(h') \]
Given \( h : g \rightarrow g' \), \( h' : g' \rightarrow g'' \) we compose these morphisms and draw them vertically.

How does the interchange law follow?

\[ (ff') \cdot (gg') = (fg)(f'g') \]

It follows from the fact that composition is a homomorphism.

* This is composition of morphisms in \( \mathcal{C} \).