Categorified Gauge Theory

\[ F = E \wedge dt + B \]

\[ \begin{array}{c}
\text{space/space part} \\
\text{space/time part}
\end{array} \]

categorification: replace identities w/ isomorphisms
- Lie group - 1 object w/ many morphisms from it to itself that are invertible
- kers - all morp. whose source is identity in Lo, namely 0.

Internalization

Take the defn. of group, and write it using commut. diagrams.

- a set \( G \)
- a function \( m: G \times G \rightarrow G \)
- an identity element \( 1 \in G \)

so we think of this as \( i: 1 \rightarrow G \) (morphism)

- terminal obj in \( \text{Set} \)
- \( (1 \text{- elt set}) \)
- "walking element"
In category theory: "\( \forall x. f(x) = g(x) \)" isn't allowed but "\( f = g \)" is.

Also okay: \( \forall x, f \circ x = g \circ x \) where
\[
A \xrightarrow{f} B \xrightarrow{g} C
\]

So "\( f = g \)" \( \iff \) \( \forall x, f \circ x = g \circ x \) where \( x \) is a generalized elt.

(including \( 1 : B \rightarrow B \) "universal elt")

(Note: products and terminal objects are both products! (binary & nullary))

Prop: A category has all products iff it has binary products and nullary products, i.e., terminal objects.

Identity of the group = multiplying no things

\[ \text{con't defn of gap} \]

\[ \text{inverses inv: } G \times G \rightarrow G \]

\[ \text{st. assoc law: } G \times (G 	imes G) \xrightarrow{m \times 1} G 	imes G \]
\[
\begin{array}{c}
\downarrow m \\
G \times G \xrightarrow{m} G
\end{array}
\]
**l/r unit laws:**

\[
G \cong G \times 1 \xrightarrow{1 \times i} G \times G \xrightarrow{m} G
\]

right unit law

- Product of object w/ terminal object is iso to object.

**inverse law:**

\[
\Delta : G \xrightarrow{\Delta} G \times G \xrightarrow{\text{inv} \times 1} G \times G
\]

\[
\Delta = \text{duplication} \quad \epsilon \quad \downarrow 1 \quad i \quad \rightarrow \quad G
\]

- send \(G\) into terminal object (counit)

\[
G \times G \xrightarrow{\text{inv} \times 1} G \times G \xrightarrow{m} G
\]

\[
\Delta : G \xrightarrow{\epsilon} 1 \xrightarrow{i} G
\]

\[
\text{comult.}
\]

\[
\text{mu ct.}
\]
Ex). Cat. of Hilbert spaces has a tensor product but aren't able to duplicate:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{H} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{H} & \otimes & \mathcal{H}
\end{array}
\]

and we aren't able to delete:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{H} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{H} & \otimes & \mathcal{H}
\end{array}
\]

So - we can define a group object in any category w/ finite products.

Want to show \((g^{-1})^{-1} = g\), i.e. show diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{1} & G \\
\downarrow \text{inv} & & \downarrow \text{inv} \\
G & \xrightarrow{1} & G
\end{array}
\]