Spring Quarter 2003

3/31/03

I. Categorified gauge theory — generalized gauge theory

the mathematics used to describe 4 forces

· involves connections of bundles

gauge theory also involves other mathematics:

live in category of sets

· groups $\xrightarrow{\text{replace w/}}$ 2-grps

· Lie grps $\longrightarrow$ Lie 2-grps

Lives in category of manifolds

| groups | sets | functs |
|---|---|---|
| ↓ | ↓ | ↓ |
| Lie grps | manifolds | smooth functs |

· Lie algebras — technical tool for studying Lie grps

↓

Lie 2-alg

We have a functor L: Lie grps $\longrightarrow$ Lie algs

takes Lie grp homo to Lie alg homo.

want a 2-functor bet. the 2-categories of Lie 2-grps & Lie 2-algs.

In ordinary gauge theory, we have
connections on bundles ⟶ 2connections on 2-bundles

connections – allow us to parallel transport arrows around

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## II. Physics

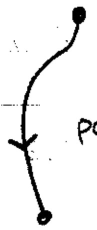
- Yang-Mills Theory – an example of a gauge theory (describes electroweak & strong forces) in Standard Model NOT gravity
- General Relativity – describes gravity a "kind of gauge theory"

describe forces not matter

Yang-Mills Theory – a gauge theory, so a "supped up" version of Y.M Theory should exist which uses all our categorified math
2- Yang-Mills Theory!

## Gauge Theory:

Moving particles



point particle

path – gives a group elt $g \in G$

In gauge theory, a particle moves — there's a group acting on our particle

Standard Model: $G = \underbrace{SU(3)}_{\text{strong}} \times \underbrace{SU(2) \times U(1)}_{\text{electroweak}}$

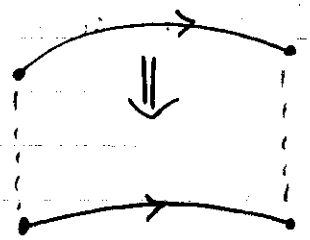

gives $gh \in G$

gives $g^{-1} \in G$.

## Categorified Gauge Theory

· replace "set" w/ "category"

* physical meaning: replace point particles by strings!



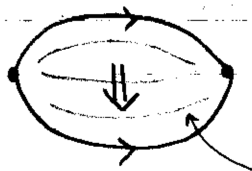
strings

We can then move these strings around in space

Now – moving things around correspond to properties of 2-grps.

category: objects - things
morphisms - way to go between things

globular picture - move string while keeping ends fixed!



this gives a morphism in our 2-grp.

path of paths.

We get a:
1-parameter family of paths

pt particles

(a category)

g an object in your 2-grp

describe how it changes - give an object in our 2-grp

## III. The Algebraic design of Physics

- Clifford algebras (assoc. algebras) & Spinors
  describe matter
  (spin-1/2 particles - matter) leptons & quarks

- Higgs-Boson (spin-0 particle — only one in Standard Model

- representations of simple Lie groups
  ex) SU(3), SU(2) are simple Lie grps SU(n)

- Jordan algebras - algebras of observables in QM

  n×n Hermitian complex matrices
  w/ product $X \cdot Y = XY + YX$

* Similar to bracket in Lie algebra! $[x,y] = xy - yx$

# Clifford Algebras

Assume spacetime is a v. space, $V$ over $\mathbb{R}$,
(in QM spacetime is a manifold)
with a "metric" –

a symmetric, bilinear map $g: V \times V \longrightarrow \mathbb{R}$
$(v,w) \longmapsto g(v,w)$

(generalized "dot product")
(usually use dot product w/ space)

If $e_i \in V$ is a basis for $V$, we define the matrix

$$g_{ij} = g(e_i, e_j)$$

$ij^{th}$ entry

Usually we want our metric $g$ to be non-degenerate
i.e. any of the following three:

1) If $g(v,w) = 0 \ \forall \ w \in V \Rightarrow v = 0$.
(true for usual dot product)

2) The map $V \longrightarrow V^*$ is 1-1.
$v \longmapsto g(v,-)$

a linear functional on $V$, waiting to eat a vector $w$.

(and hence onto when $V$ is finite dim'l)
We'll assume $V$ is finite dim'l.

3) The matrix $g_{ij}$ is invertible, and we call the inverse by $g^{ij}$,

$$\left(\sum_j\right) g_{ij}\, g^{jk} = \delta_i^k$$

Einstein's summation convention. (sum over repeated index)

and

$$g^{ij} g_{jk} = \delta_k^i$$

Examples:

① "Space" (n-dim'l) $V = \mathbb{R}^n$ ; $g(v,w) = v \cdot w$ usual dot product

② "Minkowski spacetime" (n+1)-dim'l $V = \mathbb{R}^{n+1}$

$$g(v,w) = \underbrace{v_1 w_1 + \dots + v_n w_n}_{\text{n-space coords}} - \underbrace{v_{n+1} w_{n+1}}_{\text{time coord}} \quad (GR)$$

or minus this, which is what particle physicists use.

$$g_{ij} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}$$

Both of these examples are non-degenerate.

Thm: If $g$ is a (not necessarily nondegenerate) metric on a finite-dim'l $V$, then we can find a basis $e_i$ of $V$ st

$$g_{ij} = \begin{pmatrix} \overbrace{+1 \; \ddots \; +1}^{p} & & & 0 \\ & \overbrace{-1 \; \ddots \; -1}^{q} & & \\ & & \overbrace{0 \; \ddots \; 0}^{r} & \\ 0 & & & \end{pmatrix}$$

It takes 3 #'s to specify symmetric, bilinear form:

$p$ = # $+1$'s

$q$ = # $-1$'s

$r$ = $0$

$p + q + r = \dim V$

Laplacian $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ minus this is $\Delta$.

Wave operator or Box oper. $\square = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

But — $g$ is nondegenerate iff $r=0$.

Corollary: If $(V,g)$ is a v.space w/ a nondegenerate metric $g$, then $(V,g) \cong \mathbb{R}^{p,q}$ where

$$\mathbb{R}^{p,q} := \mathbb{R}^{p+q} \text{ w/ metric } g_{p,q} \text{ whose matrix is}$$

$$\begin{array}{c c} & \\ p & \\ q & \\ & \end{array}\left(\begin{array}{ccc|ccc} +1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & +1 & & & \\ \hline & & & -1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & -1 \end{array}\right)$$
$$\qquad\quad p \qquad\qquad q$$

$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

We're interested in $p=0$ or $1$ or $q=0$ or $1$.

In special relativity, we live in either $\mathbb{R}^{3,1}$ (if you do GR) or $\mathbb{R}^{1,3}$ (if you do particle physics)

"Minkowski spacetime"

$\vec{\nabla}\cdot\vec{\nabla} = \nabla^2$

$-\nabla^2 = \Delta$

We can define a Laplace operator on functions

$$\psi : V \longrightarrow \mathbb{R} \quad \text{scalar field}$$

(assigns to each vector a real #)

Laplacian is 3-dim'l, so a triangle

by:

$$\square = g^{ij}\partial_i\partial_j \quad \text{where}$$

$\partial_i = \frac{\partial}{\partial x^i}$ and $x^i$ are coords. on $V$ corresponding to the basis $e_i$ for which $g_{ij} = g(e_i, e_j)$

* In fact, $\square$ is independent of the basis $e_i$!

$1/\text{superscript}$ = subscript in Einstein sum convention.

coeffs of basis elts.

Want forms to have superscripts $dx^i$. $\qquad X = x^1 e_1 + x^2 e_2 + \dots + x^n e_n$

If $(V, g) = \mathbb{R}^{p,q}$ then

$$g_{ij} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

and

so

$$g^{ij} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

$$\Box = \partial_1^2 + \dots + \partial_p^2 - \partial_{p+1}^2 - \dots - \partial_{p+q}^2$$

If $q = 0$, then $\Box = \nabla^2$ $\qquad \mathbb{R}^{p,0}$ is "space".

$$= \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \quad \text{of } \mathbb{R}^{1,3}$$

"spacetime"

$\Box$ - describes waves moving @ speed 1.

The <u>wave equation</u>: $\Box \psi = 0$.

describes scalar waves moving at speed 1, e.g.:

$$\psi(t, x, y, z) = \sin(\omega(t - x))$$

waves moving in $x$ direction @ speed 1.

differentiate $\psi$ twice - bring out $\omega$ twice, get -sine

w/r/t $t$

w/r/t $x$ - get negative twice.

More generally

$\psi(t, \underbrace{x, y, z}_{\vec{x}}) = f(t - \vec{x} \cdot \vec{k})$ $\quad$ $\vec{k} \in \mathbb{R}^3$ is a unit vector

is a solution describing waves moving at speed 1 in $\vec{k}$ direction.

Thm.: Every solution of $\square \psi = 0$ is a linear comb. of solns of above type.

(thm needs to be made more precise — is "f" above cont. or smooth?)

Dirac: Find an differential operator $\not{\partial}$ such that $\not{\partial}^2 = \square$.
i.e. Find a square root of the Laplace operator.

You can't do this for

$$\square : C^\infty(V, \mathbb{R}) \longrightarrow C^\infty(V, \mathbb{R}).$$

(Here, $S = \mathbb{R}$ and $V = \mathbb{R}$)

unless $(V, g) = \mathbb{R}^{1,0}$ i.e. $\square = \frac{\partial^2}{\partial x^2}$ has a square root
We let $\not{\partial} = \frac{\partial}{\partial x}$

Note: can't do it for $\mathbb{R}^{0,1}$ since we'd get $\not{\partial} = i \frac{\partial}{\partial x}$
which acts on complex NOT real valued functions.

* We can use Laplacian on functs taking values in a v. space — just apply to each component.

We'll solve Dirac's question for

$\Box : C^\infty(V,S) \longrightarrow C^\infty(V,S)$ for some real vector space $S$.

(we just differentiate each component)

Let's try $\not{\partial} = \gamma^i \partial_i$ where $\gamma^i : S \longrightarrow S$ is a linear transf of $S$.

we're summing over $i$.

(Can have different maps $\forall\, i$)

Think of w/ a basis for $S$ — $\gamma^i$'s are <u>matrices</u>.

<u>Defn</u>: $(\not{\partial}\psi)(x) := \sum_i \gamma^i (\underbrace{\partial_i \psi(x)}_{\in S})$

$\psi \in C^\infty(V,S)$
$x \in V$
$\psi(x) \in S$, so
$\partial_i \psi(x) \in S$

When does this trick work? When is $\not{\partial}^2 = \Box$?

$$\not{\partial}^2 = (\gamma^i \partial_i)^2 = \sum_{i,j} (\gamma^i \partial_i)(\gamma^j \partial_j)$$

$$= \sum_{i,j} \gamma^i \gamma^j \partial_i \partial_j$$

symmetrize

mixed partials commute, so $\partial_i \partial_j$ is symmetric under $i \,\&\, j$.

$$(*) \quad = \sum_{i,j} \tfrac{1}{2} \underbrace{(\gamma^i \gamma^j + \gamma^j \gamma^i)}_{\text{a linear transf.}} \partial_i \partial_j$$

$$= \sum_{i,j} \underbrace{g^{ij}}_{\text{a number}} \partial_i \partial_j = \Box \quad \text{box operator}$$

Want to say $g^{ij} = \gamma^i \gamma^j + \gamma^j \gamma^i$, but not the same thing!

* **Fact**: If $A_{ij} = A_{ji}$ then

$$B^{ij} A_{ij} = \frac{1}{2}\left(B^{ij} + B^{ji}\right) A_{ij}$$

But — we can think of $g^{ij}$, which is a number, as a linear transformation — the linear transf. which multiplies everything by $g^{ij}$.

Naively — we could have skipped step (*) and could have gotten the trick to work w/

$$\gamma^i \gamma^j = g^{ij} I \qquad I : S \to S \text{ is identity linear transf.}$$

But $g^{ij} = g^{ji}$, so this would force

$$\gamma^i \gamma^j = \gamma^j \gamma^i.$$

But — doing (*) the trick also works if

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2 g^{ij} I \quad \text{and}$$

LHS is obviously symmetric under $i \leftrightarrow j$.

We'll solve the above eqn by thinking of $\gamma^i$'s not as linear transf, but as elts of an algebra.

Note: $\boxed{\text{algebra}}$ — a ring & v. space where '+' in v. space = '+' in ring
and our algebras always have a unit.

Defn: Given a v. space w/ metric $(V, g)$ let the
Clifford algebra $C(V,g)$ be the assoc. algebra (over $\mathbb{R}$)
generated by $V$ (meaning take linear combs of elts of $V$, where addition in $C$ is same as in $V$)
modulo the relations

$$vw + wv = 2g(v,w)\cdot 1 \quad \text{for } v, w \in V$$

Prop: If $(V,g)$ has basis $e_i$, then $C(V,g)$ is isomorphic to the algebra generated by the elts $e_i$ mod the relns

$$e_i e_j + e_j e_i = 2g_{ij}\cdot 1$$

products of linear combs of basis elts.

Each elt in $V$ is a linear comb. of basis elts, so this is enough to get $C(V,g)$.
First we take linear combs of $e_i$'s — this gives us $V$.
Then multiply them, mod these relations, and get $C(V,g)$.

Prop: Suppose $\gamma : C(V,g) \longrightarrow \mathrm{End}(S)$ is an
algebra homomorphism where $S$ is a real v. space
and $\mathrm{End}(S)$ is the alg. of all lin. transf. of $S$.

Then, letting $\gamma_i = \gamma(e_i)$ we get

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2g_{ij} I$$

($\gamma$ preserves '+' and multiplication)

We call an alg. homo a "representation" on an alg.

So – finding solns. to $\cancel{\partial}^2 = \Box$ amounts to

1) understanding $C(V, g)$
(these are nice – like $\mathbb{C}$, $\mathbb{H}$, $4\times4$ $\mathbb{C}$ matrices)

2) understanding their representations

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The algebra generated by v. space $V$ means:

Take all formal linear combs. of all formal products of elts. of $V$.

Fact: Let $A$ be an $n \times n$ matrix. Then

$$x^T A x = \sum_{i,j=1}^{n} a_{ij}\, x_i x_j = \sum_{i,j=1}^{n} \tfrac{1}{2}(a_{ij} + a_{ji})\, x_i x_j$$

Ex) $n = 2$

$$x^T A x = (x_1\ x_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 a_{11} x_1 + x_1 a_{12} x_2 + x_2 a_{21} x_1 + x_2 a_{22} x_2$$

$$= \sum_{i,j=1}^{2} a_{ij}\, x_i x_j$$

$$= \sum_{i,j=1}^{2} \tfrac{1}{2}(a_{ij} + a_{ji})\, x_i x_j$$

$$= \tfrac{1}{2}(a_{11} + a_{11}) x_1 x_1 + \tfrac{1}{2}(\underline{a_{21}} + \underline{\underline{a_{12}}}) x_2 x_1 + \tfrac{1}{2}(\underline{\underline{a_{12}}} + \underline{a_{21}}) x_1 x_2 + \tfrac{1}{2}(a_{22} + a_{22}) x_2 x_2 \quad \checkmark$$

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proof of fact on prev pg:

$$\sum_{i,j=1}^{2} B^{ij} A_{ij} = B^{11}A_{11} + B^{12}A_{12} + B^{21}A_{21} + B^{22}A_{22}$$

$$\sum_{i,j=1}^{2} \tfrac{1}{2}(B^{ij} + B^{ji}) A_{ij} = \tfrac{1}{2}(B^{11} + B^{11}) A_{11} + \tfrac{1}{2}(B^{12} + B^{21}) A_{12} + \tfrac{1}{2}(B^{21} + B^{12}) A_{21} + \tfrac{1}{2}(B^{22} + B^{22}) A_{22}$$

$$= B^{11}A_{11} + B^{22}A_{22} + \underline{\tfrac{1}{2} B^{12} A_{12}} + \underline{\underline{\tfrac{1}{2} B^{21} A_{12}}} + \underline{\underline{\tfrac{1}{2} B^{21} A_{21}}} + \underline{\tfrac{1}{2} B^{12} A_{21}}$$

But $A_{12} = A_{21}$

**Note:** Product of symmetric matrices is NOT symmetric.

4/1/03

Recall from last time:

$(V, g)$ is a v.space w/ metric $g$.

V-spacetime

If $\not{\partial} = \gamma^i \partial_i$ $\quad \gamma^i \in \mathrm{End}(S)$, then

$$\not{\partial}^2 = \Box : C^\infty(V, S) \longrightarrow C^\infty(V, S) \text{ if}$$

$$(*) \qquad \gamma^i \gamma^j + \gamma^j \gamma^i = 2g^{ij}$$

If this holds, we call $\not{\partial}$ a <u>Dirac-operator</u> and we call $S$ a space of <u>spinors</u>.

Let $C(V, g)$, the <u>Clifford algebra</u> be the algebra generated by $V$: $T(V)$ modulo some relations:

$$C(V, g) = T(V) \big/ \langle vw + wv - 2g(v, w) 1 \rangle$$

tensor algebra

ideal gen. by these relns.

$$T(V) = \bigoplus_{n=0} V^{\otimes n}$$

If we have a rep

$$\gamma : C(V, g) \longrightarrow \mathrm{End}(S)$$

and $e_i \in V$ is a basis of $V$ and let

$$\gamma(e_i) = \gamma_i$$

then we get $\gamma_i \gamma_j + \gamma_j \gamma_i = 2g_{ij}$

(Notice our superscripts on prev pg have changed to subscripts!)

<u>Note</u>: Since $g$ is nondegenerate the map

$$v \longmapsto g(v, -)$$

is an iso. from $V$ to $V^*$, which lets us raise and lower indices, so this solves our original <u>problem</u>: Let $\boxed{\gamma^i = g^{ij}\gamma_j}$

Since $g$ is invertible, we have $g^{ij}$ & $g_{ij}$

stick this into (*) on prev pg & we get above eqn.

<u>Note</u>: If $\psi : V \longrightarrow S$ satisfies the massless <u>Dirac eqn</u>

$$\not{\partial}\psi = 0$$

then $\Box\psi = \not{\partial}^2\psi = 0$

Physically — solutions of this Dirac eqn. describe massless spin-½ particles.

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Problem: Understand $C(V,g)$ & its representations.
Suffices to consider

$$(V, g) = \mathbb{R}^{p,q} \qquad \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} \end{array}\right) \; g_{p,q}$$

Recall— $g_{p,q}$ is the matrix w/ $p$ $+1$'s, $q$ $-1$'s on diagonal.

So when $i \neq j$ we're not on diagonal, so zero.

when $i=j$ we get $\pm 1$ depending on where we are.

Let
$$C_{p,q} = C(\mathbb{R}^{p,q}).$$

Concretely, $C_{p,q}$ is the algebra generated by $e_1, \ldots, e_p, e_{p+1}, \ldots, e_q \in \mathbb{R}^{p,q}$ with the relations
$$e_i e_j + e_j e_i = 2(g_{p,q})_{ij}$$

If $i \neq j$ $\quad e_i e_j = -e_j e_i$

this is identity elt in Clifford alg. NOT $1 \in \mathbb{R}$.

If $i=j$ $\quad e_i^2 = \begin{cases} +1 & i = 1, \ldots, p \\ -1 & i = p+1, \ldots, p+q \end{cases}$

So— a Clifford alg consists of square roots of $1, -1$ which anti-commute.

<u>Summary</u>: $C_{p,q}$ is alg. gen. by

$p$ - square roots of $1$
$q$ - square roots of $-1$
all of which anticommute

$$C_{p,q}$$

p square roots of 1

q square roots of -1

| q \ p | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\mathbb{R}$ | $\mathbb{R}\oplus\mathbb{R}$ | $M_2(\mathbb{R})$ | $M_2(\mathbb{C})$ |
| 1 | $\mathbb{C}$ | $M_2(\mathbb{R})$ | $M_2(\mathbb{R})\oplus M_2(\mathbb{R})$ | $M_4(\mathbb{R})$ |
| 2 | $\mathbb{H}$ | $M_2(\mathbb{C})$ | | |
| 3 | $\mathbb{H}\oplus\mathbb{H}$ | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |

Note: $C_{0,0} = \mathbb{R}$ (algebra must have mult. unit 1)

$C_{0,1} \cong \mathbb{C}$, $i^2 = -1$

$C_{0,2}$: $i^2 = j^2 = -1$, $ij = -ji := k$

so, $k^2 = ijij = -i(-ij)j = -i^2j^2 = -(-1)(-1) = -1$

$jk = jij = -ij^2 = i$ / $kj = ijj = -jij = -i$ / $ki = iji = -i^2j = j$

$ik = iij = -j$

So, $C_{0,2} \cong \mathbb{H}$, the <u>quaternions</u>

* We started w/ 2 square roots of -1, and got another one – $k$.

<u>Guess</u>: $C_{0,3} \cong \mathbb{O}$ octonians? NO! the octonians are not associative.

$C_{1,0}$ : $i^2 = 1$

This is isomorphic to $\mathbb{R} \oplus \mathbb{R}$. (This is a coproduct should call it '$\mathbb{R} \times \mathbb{R}$')
Do algebra operations component-wise.

Why?

*

$$(1,0) \longmapsto \frac{1+i}{2}$$

$$(0,1) \longmapsto \frac{1-i}{2}$$

$\mathbb{R} \oplus \mathbb{R}$

Check this is a homomorphism since $(1,0)$, $(0,1)$ $\frac{1+i}{2}$, $\frac{1-i}{2}$ are idempotent i.e. $x^2 = x$.

These are called the "dual numbers".

Check this is onto — everything in $C_{1,0}$ is a linear comb of $1$ & $i$.
They have the same dimension: 2, so it's 1-1.
$\dim(\mathbb{R} \oplus \mathbb{R}) = 2$
$\dim(C_{1,0}) = 2$ basis $\{1, i\}$.

$\begin{cases} \mathbb{C} - \text{Laplace eqn} \\ \mathbb{R} \oplus \mathbb{R} - \text{Heat eqn} \end{cases}$

To continue finding these — we'll consider Pauli matrices.

$C_{3,0}$ – alg gen. by 3 anti-commuting square roots of $-1$ w/ no other relns.

The Pauli matrices are ½ times:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

<u>Note</u>: $I^2 = J^2 = K^2 = 1$ and they anticommute.

Thus, there's a homo:

$$\alpha : C_{3,0} \longrightarrow M_2(\mathbb{C}) \qquad 2\times 2\ \mathbb{C} \text{ matrices}$$

Is $\alpha$ an isomorphism?

($M_2(\mathbb{C})$ is 8-dim'l) Check: $\alpha$ is <u>onto</u> because $I, J, K$ generate $M_2(\mathbb{C})$.

Check: $\alpha$ is 1-1 means checking $I, J, K$ satisfy only relns following from those above ($I^2 = J^2 = K^2 = 1$ a; $IJ = -$ etc)

This is hard! So, instead we'll show

$$\dim(M_2(\mathbb{C})) = \dim(C_{3,0})$$

$\parallel$

8    4 places in matrix, 2 choices $\forall$ place: $1, i$

<u>Note</u>: If $\dim(V) = \dim(W)$ then $f: V \to W$ is 1-1 iff $f$ is onto.

proof – later on.

Prop: $\dim(C_{p,q}) = 2^{p+q}$

pf: $C_{p,q}$ has a basis

$$\begin{array}{l} 1 \\ e_i \\ e_i e_j \quad (i<j) \\ e_i e_j e_k \quad (i<j<k) \\ \vdots \end{array}$$

Note: $e_i e_j + e_j e_i = g_{ij} 1$ this reln. lets us "straighten out" a product of $e_i$'s.

So, $\dim(C_{p,q}) = 2^{p+q}$. ∎

Note: $I$ & $J$ generate a subalgebra of $M_2(\mathbb{C})$ which is isomorphic to $C_{2,0}$ and is contained in $M_2(\mathbb{R})$. (since $I, J \in M_2(\mathbb{R})$) and is 4-dim'l.
Since $\dim(M_2(\mathbb{R})) = 4$, we get $C_{2,0} \cong M_2(\mathbb{R})$

Note – Keeping just $I$ – gives us $\mathbb{R}\oplus\mathbb{R}$
$I$ gives any diagonal matrix.

Similarly – $I$ generates a subalg. of $M_2(\mathbb{R})$ consisting of diagonal matrices; this alg is $\mathbb{R}\oplus\mathbb{R}$.

Quest: What's $C_{1,1}$? Answer: $M_2(\mathbb{R})$.

It's 4-dim'l and is gen. by $i, j$ w/ $i^2 = 1$, $j^2 = -1$, $ij = -ji$

Note — Just because $M_2(\mathbb{R})$ is gen. by 2 square roots of 1 doesn't mean it can't be generated by a square root of 1 & a square root of -1.

In fact, $C_{1,1} \cong M_2(\mathbb{R})$ w/

$$i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

this is how we represent the complex # $i \in \mathbb{C}$ as a rotation of the plane.

Check: $ij = -ji$
$i^2 = 1$, $j^2 = -1$.

This gives a homo of
$C_{1,1} \longrightarrow M_2(\mathbb{R})$

just check that gen. & relns are sent correctly.

$\dim(C_{1,1}) = 4$
$\dim(M_2(\mathbb{R})) = 4$

So, once we show $i, j$ generate $M_2(\mathbb{R})$, we get an iso.

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If $A, B$ are algebras, then $A \oplus B$ (should write as $A \times B$) and $A \otimes B$ are algebras.

$A \oplus B$ becomes an algebra w/

$$(a,b)(a',b') = (aa', bb')$$

$A \otimes B$ becomes an algebra w/

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

**Example**: Let $A$ be an algebra over $\mathbb{R}$, then

$$A \otimes M_n(\mathbb{R}) \cong M_n(A)$$

$$a \otimes E_{ij} \longmapsto aE_{ij}$$

1 in $ij^{th}$ entry
zeros elsewhere

Every matrix in $M_n(\mathbb{R})$ is a linear comb. of $E_{ij}$'s.

**Prop**: $C_{p,q} \otimes C_{1,1} \cong C_{p+1,q+1}$

$C_{2,1}$ $*$ So — $(\mathbb{R}\oplus\mathbb{R}) \otimes M_2(\mathbb{R}) = M_2(\mathbb{R}\oplus\mathbb{R})$

$M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \qquad = M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$

$\qquad ||$

$C_{3,1}$: $*$ $M_2\big(M_2(\mathbb{R})\big)$ — $2\times 2$ matrices whose entries are $2\times 2$ matrices!

$\begin{array}{|c|c|}\hline + & + \\ \hline + & + \\ \hline\end{array}$

$\qquad ||$

$M_4(\mathbb{R})$

__proof of Prop:__ Consider $C_{p,q}$ and $C_{1,1}$ where

$C_{p,q}$ is generated by $e_i$ $(i = 1, \ldots, p+q)$ and $C_{1,1}$ is generated by $i, j$ w/ $i^2 = 1$, $j^2 = -1$

Consider these elts of $C_{p,q} \otimes C_{1,1}$:

$e_i \otimes i$
$1 \otimes j$
$1 \otimes ij$

$\dim(V \otimes W) = \dim(V) \cdot \dim(W)$

* To see these anticommute: look at 1st 2: $e_i$ commutes w/ 1, but $i$ & $j$ anticommute.

Ex) $(e_i \otimes i)(1 \otimes ij) = e_i \otimes iij = e_i \otimes j$

$(1 \otimes ij)(e_i \otimes i) = e_i \otimes iji = -e_i \otimes iij = -e_i \otimes j$

$e_i$'s always anticommute w/ each other. $i, j$ anticommute.

* They are all square roots of $\pm 1$.

$(e_i \otimes i)^2 = (e_i^2 \otimes i^2) = e_i^2 \otimes 1 = \pm 1 \otimes 1$

$1 \otimes 1$ is mult. iden in tensor product

the "1" in $C_{p,q} \otimes C_{1,1}$.

$(1 \otimes j)^2 = 1 \otimes j^2 = 1 \otimes -1 = -1 \otimes 1$

$(1 \otimes ij)^2 = 1 \otimes ijij = -1 \otimes i^2 j^2 = -1 \otimes -1 = 1 \otimes 1$

$(1 \otimes j)$ is a extra square root of $-1$
$(1 \otimes ij)$ is an extra square root of $1$.

so we end up in $C_{p+1,q+1}$ 1 more root of $-1$, 1 each

This lets us get a homo:

$$C_{p+1,q+1} \longrightarrow C_{p,q} \otimes C_{1,1}$$

for it to be an iso, check 1) it's onto

2) have same dim.

$\dim(C_{p+1,q+1}) = 2^{p+q+2}$

$\dim(C_{p,q} \otimes C_{1,1}) = 2^{p+q} \cdot 2^2 = 2^{p+q+2}$

1) Want to get $1 \otimes i = (1 \otimes ij)(1 \otimes j)$

Want to get linear combs – mult by real scalars.

Want to get everything in $C_{p,q} \otimes C_{1,1}$ as a lin comb. of these 3 guys.

For ontoness – we need to check these special elts generate $C_{p,q} \otimes C_{1,1}$. $\blacksquare$

① **Prop:** $C_{p,q} \otimes C_{2,0} \cong C_{q+2,p}$

② **Prop:** $C_{p,q} \otimes C_{0,2} \cong C_{q,p+2}$

proof of both above is exactly like the proof of the previous one, except we need to check how many square roots of $1$ & $-1$ we get.

In proof of ① $C_{2,0}$ has 2 different square roots of 1.

$C_{2,0}$ is gen. by $i$ & $j$ w/ $i^2 = j^2 = 1$.

Want sq. roots of 1 to turn into sq. roots of -1.

① **We'll use** $e_i \otimes ij,\ 1 \otimes i,\ 1 \otimes j$

$$(e_i \otimes ij)^2 = e_i^2 \otimes ijij = -e_i^2 \otimes i^2 j^2 = -e_i^2 \otimes 1$$

$$(1 \otimes i)^2 = 1 \otimes i^2 = 1 \otimes 1$$

$$(1 \otimes j)^2 = 1 \otimes j^2 = 1 \otimes 1$$

switch sqr root of 1 to $-1$.

② We'll use: same elts, but $i^2 = j^2 = -1$

$(e_i \otimes ij)^2 = e_i^2 \otimes ijij = -e_i^2 \otimes i^2 j^2 = -e_i^2 \otimes 1$

$(1 \otimes i)^2 = 1 \otimes i^2 = -1 \otimes 1$

$(1 \otimes j)^2 = 1 \otimes j^2 = -1 \otimes 1$

now sq. roots of $-1$

Ex) $C_{0,3} = (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} = \mathbb{H} \oplus \mathbb{H}$, but $\mathbb{H}$ is an alg over $\mathbb{R}$ so $\mathbb{H} \otimes \mathbb{R} \cong \mathbb{H}$

tensor prod. dist. over sums.

$C_{1,0} \otimes C_{0,2} \cong C_{0,3}$

To fill out rest of table — need to know mult. table for $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

These are division algebras — every non-zero elt has mult. inverse.

Division algebra mult. table:

Thm: The only assoc. algs in which $xy = 0 \Rightarrow x = 0$ or $y = 0$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. (the only division algs)

These are called div. algs.

| $\otimes$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ |
|---|---|---|---|
| $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ |
| $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C} \oplus \mathbb{C}$ | $M_2(\mathbb{C})$ |
| $\mathbb{H}$ | $\mathbb{H}$ | $M_2(\mathbb{C})$ | $M_4(\mathbb{R})$ |

$\mathbb{C} \otimes \mathbb{C}$ is 4-dim'l, commut.

* $\left[\begin{array}{l}\mathbb{C}\oplus\mathbb{C}\cong\mathbb{C}\otimes\mathbb{C} \quad \text{categorification of } 2+2=2\times 2\\ \dim\mathbb{C}=2\end{array}\right.$

<u>Prop:</u> $\mathbb{C}\otimes\mathbb{H}\cong M_2(\mathbb{C})$

pf. $\mathbb{C}\otimes\mathbb{H}=C_{0,1}\otimes C_{0,2}\cong C_{1,2}\cong M_2(\mathbb{C})$

<u>Prop:</u> $\mathbb{H}\otimes\mathbb{H}\cong M_4(\mathbb{R})$

pf. $\mathbb{H}\otimes\mathbb{H}=C_{0,2}\otimes C_{0,2}\cong C_{2,2}\cong M_4(\mathbb{R})$

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Facts:

① $\mathbb{C}$ is commut.

$$\begin{aligned}(x+iy)+(a+ib)&=(x+a+(b+y)i)\\&=a+x+(y+b)i\\&=(a+ib)+(x+iy)\end{aligned}$$

$$\begin{aligned}(x+iy)(a+ib)&=ax+xib+iya+iyib\\&=ax-yb+i(xb+ya)\end{aligned}$$

$$\begin{aligned}(a+ib)(x+iy)&=ax+ibx+aiy+ibiy\\&=ax-by+i(bx+ay)\quad\checkmark\end{aligned}$$

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<u>Facts:</u> ① $A\otimes\mathbb{R}[n]\cong A[n]$

nxn matrices w/ $\mathbb{R}$ entries

② $\mathbb{R}[n]\otimes\mathbb{R}[m]\cong\mathbb{R}[nm]$

② Let $\dim V \overset{n}{=} \dim W$. Then let $f: V \longrightarrow W$ be a linear transf.

a) $f$ is 1-1 $\Rightarrow$ $f$ is onto.

We have $\dim V = \dim \ker f + \dim \operatorname{im} f$

$f$ is 1-1 $\Rightarrow$ $\ker f = \{0\}$

So we have $n = \dim(\operatorname{im} f)$

But any 2 finite dim'l v. space w/ same dimension are isomorphic. $\Rightarrow$ $\operatorname{im} f = W$.

b) $f$ is onto $\Rightarrow$ $f$ is 1-1.     $f$ onto $\Rightarrow$ $\operatorname{im} f = W$

$$\begin{aligned}\dim V &= \dim \ker f + \dim \operatorname{im} f\\ n &= \dim \ker f + \dim W\\ n &= \dim \ker f + n\\ 0 &= \dim \ker f\end{aligned}$$

$\Rightarrow$ $\ker f = \{0\}$, so $f$ is 1-1.