4/7/03

## Category Theory:

category: things, homomorphisms bet those things

- in many categories – we can "combine" 2 objects

  ex) $V \otimes W$ or $V \oplus W$ for $V, W$ v. spaces $\otimes, \oplus$

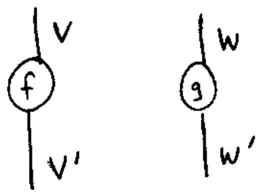
  gives monoidal category.

<u>But</u> – we want to do things for morphisms as well!

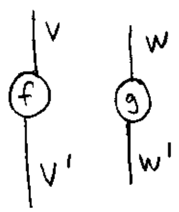
| <u>objects</u> | <u>morphisms</u> |
|---|---|
| $V, W \in \mathrm{Vect}$ give $V \otimes W \in \mathrm{Vect}$ | $f: V \longrightarrow V'$, $g: W \longrightarrow W'$ linear maps give $f \otimes g: V \otimes W \longrightarrow V' \otimes W'$, $f \otimes g(v \otimes w) = f(v) \otimes g(w)$ |

circuits: parallel & series

We draw morphisms as:

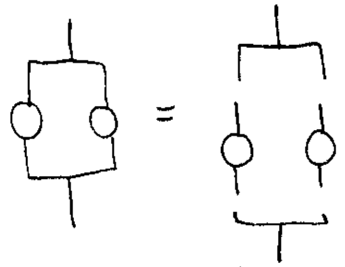



give




"parallel"

circuits:




(electrical engineering)

## Monoids

**Defn:** A <u>monoid</u> is a set $M$ w/ an assoc. binary operation called multiplication $\cdot$ together w/ a left/right unit $1 \in M$.


Ex) $(\mathbb{N}, +)$

We'll now replace the word "set" by other words, to transpose this idea into other categories.

So — we'll rewrite above defn. as

A monoid is a <u>set</u> $M$ together w/ a <u>function</u> $\cdot : M \times M \longrightarrow M$ and a <u>function</u> $i : 1 \longrightarrow M$

$1$ — the 1-elt set $1 = \{*\}$.

so $i(*) = 1 \in M$

* We've taken the unit $1$ above and turned it into the function $i$. *

such that (write out conditions in terms of commut. diagrams)

$$\begin{tikzcd}
M \times M \times M \arrow[r, "\cdot \times 1_M"] \arrow[d, "1_M \times \cdot"'] \arrow[dr, phantom, "\circlearrowright"] & M \times M \arrow[d, "\cdot"] \\
M \times M \arrow[r, "\cdot"'] & M
\end{tikzcd}$$

commutes (this is assoc. law)

the 1-elt set

$$\begin{tikzcd}
M \arrow[r, "\sim"] \arrow[d, "1_M"'] & M \times 1 \arrow[d, "1_M \times i"] \\
M & M \times M \arrow[l, "\cdot"]
\end{tikzcd}$$

Commutes $(x \cdot 1_m = x)$
left unit law

and corresponding right unit law:

1-elt set

$$\begin{tikzcd}
1 \times M \arrow[d, "i \times 1_M"'] & M \arrow[l, "\sim"'] \arrow[d, "1_M"] \\
M \times M & M \arrow[l, "\cdot"]
\end{tikzcd}$$

Commutes
(right unit law)

This defn allows us to generalize a monoid to other categories!

**monoid object:** We use this defn to internalize the defn of monoid — replace <u>set</u> & <u>function</u> by <u>object of C</u> and <u>morphism of C</u>

where C is any category w/ finite products
(or - more generally, in <u>any</u> monoidal category)

<u>Note</u>: Not true that terminal obj. is the unit object for product.

Ex) In Vect — terminal obj. is 1-dim'l v. space and unit for $\otimes$ is ground field, $k$

Ex) If C is Vect, a monoid object in Vect is an (associative) algebra.

Ex) If C is Top, a monoid object in Top is a topological monoid.
(like topological grp)

$1$ = 1-pt set

Category$^{op}$ — formally turn all arrows around. all objects remain the same.

$f: x \to y$ in $C^{op}$ is $f: y \to x$ in $C$  $f \circ g$ in $C^{op}$ is $g \circ f$ in $C$

Ex) If $C = \text{Vect}^{op}$, a monoid in $C$ is a coalgebra.

Note — Doing op twice gets us back where we've started.

X is whatever: v.space, set

Defn: A co-X in C is an X in $C^{op}$.

Ex) In $C = \text{Set}^{op}$, a monoid in C is a co-monoid.

* Try: Classify all comonoids.

Ex) In C = Cat — category of all small categories.
(Problem: set of all sets that contain themselves — paradox)
a monoid in C is a strict monoidal category.
The good ones are the weak ones!

Defn: A strict monoidal category consists of a category M, a functor

$$\otimes : M \times M \longrightarrow M$$

a functor

$$i : \underline{I} \longrightarrow M$$

the terminal category (terminal obj)

I is the category w/ 1-elt $x$ and 1-morphism: $1_x$

This is the unit for $\otimes$.

$i(I) = 1 \in M.$

s.t. $\otimes$ is associative & left & right unit laws hold. i.e. diagrams on prev pg commute.

Thus: $(x \otimes y) \otimes z = x \otimes (y \otimes z) \quad x, y, z \in M$

$$1 \otimes x = x = x \otimes 1$$

But - in most categories these are only isomorphic!

Ex) In Vect, $(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$
and
$$\mathbb{C} \otimes V \cong V$$

Defn: In practice — we should weaken this defn. so that these eqns become isos:

associator: $\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$

left/right units $\begin{cases} l_x : 1 \otimes x \xrightarrow{\sim} x \\ r_x : x \otimes 1 \xrightarrow{\sim} x \end{cases}$

But this still isn't good enough! It's not okay to just change eqns to isos.

In any category w/ way of gloming 2 things together: $\otimes$, etc we can form a weak monoidal category.

For example — in Vect, there are 5 ways to parenthesize 4 things:

$$\begin{tikzcd}
 & (W\otimes X)\otimes(Y\otimes Z) \arrow[dr] & \\
((W\otimes X)\otimes Y)\otimes Z \arrow[ur, "\alpha_{W\otimes X, Y, Z}"] \arrow[d, "\alpha_{W,X,Y}\otimes 1_Z"'] & & W\otimes(X\otimes(Y\otimes Z)) \\
(W\otimes(X\otimes Y))\otimes Z \arrow[rr] & & W\otimes((X\otimes Y)\otimes Z) \arrow[u]
\end{tikzcd}$$

We want a unique iso. bet things!

Just one way for them to be the same.
Otherwise — if not unique, the two things have different ways of being the same.

This diagram commutes in Vect.

So — since just adding $\alpha, l, r$ isn't enough, we also demand that the above pentagon commutes (pentagon id). We also get a diagram for $l, r$ which we demand commutes.

Then, by MacLane's coherence thm, we get:

* All diagrams built using $\otimes, \alpha, l, r$ commute.

Adding in these diagrams we get the defn. of a weak monoidal category.

weak monoidal categories — what show up in nature.

Groups are used more often than monoids — so we do above internalization to get a group object — which has a functor

$$\mathrm{inv}: G \longrightarrow G.$$

So — if we do this all for groups, we get "groups in C"
and if C = Cat, we get a "groupal category" or <u>strict 2-group</u>.

What we really want are "weak 2-groups"