4/14/03 Clifford Algebras

(V,g) - a (real, finite dim'l) vector space w/ (nondegenerate) metric.

We invented $C(V,g)$ – the alg. generated by V w/ relns.

$$vw + wv = 2g(v,w)$$

(In particular – $v^2 = g(v,v)$.)

to find a square root of the Laplace operator.

But as a spinoff, it turns out that this "Clifford algebra" is closely related to the group:

$$\underset{\text{orthogonal}}{O}(V,g) = \left\{ T : V \longrightarrow V \;\middle|\; \begin{array}{l} T \text{ linear } \& \\ g(Tv,Tw) = g(v,w) \\ \forall\, w,v \in V \end{array} \right\}$$

preserves the metric

and also to

$$SO(V,g) = \{ T \in O(V,g) \mid \det(T) = 1 \}$$

special orthogonal

clearly a subgrp since $\det(AB) = \det(A)\cdot\det(B)$

Ex) If $(V,g) = \mathbb{R}^n$ w/ its usual dot product. $g(v,w) = v\cdot w$

then we write

$$O(V,g) = O(n)$$

all reflections & rotations in n-dimensions

$O(n)$ = "n-dimensional rotation/reflection group"

and $SO(V,g) = SO(n)$ = "n-dim'l <u>rotation</u> grp"

Ex) If $(V,g) = \mathbb{R}^{p,q}$ then we write

$SO(V,g) = SO(p,q)$

then

$O(V,g) = O(p,q)$

$SO(3)$ is our familiar rotation grp and $SO(3,1)$ is our familiar Lorentz grp.

The Clifford algebra $C(V,g)$ has a subset $Pin(V,g)$ which is closed under mult. & inverses, hence a group.

(Note – Clifford algebra is an algebra – no reason why each elt should have an inverse.)

And $\underbrace{Pin(V,g)}_{\text{related to } O(V,g)}$ has a subgroup $\underbrace{Spin(V,g)}_{\text{related to } SO(V,g)}$.

So we have a commutative diagram:

$$\begin{array}{ccccc} \mathrm{Spin}_0(V,g) & & \mathrm{Spin}(V,g) & \hookrightarrow & \mathrm{Pin}(V,g) \subseteq C(V,g) \\ \downarrow {\scriptstyle 2\text{-}1 \;\&\; \text{onto}} & & \downarrow {\scriptstyle 2\text{-}1 \;\&\; \text{onto}} & & \downarrow {\scriptstyle 2\text{-}1 \;\&\; \text{onto}} \\ SO_0(V,g) & \hookrightarrow & SO(V,g) & \hookrightarrow & O(V,g) \end{array}$$

Lie grp, hence manifold, so can ask if it's connected. Usually not, so we can look @ connected component.

<u>Notation</u>: Subscript "o" means the <u>identity component</u>: i.e. the connected component containing 1.

Note: $\mathrm{Spin}(\mathbb{R}^n) = \mathrm{Spin}(n)$

$\mathrm{Spin}(\mathbb{R}^{p,q}) = \mathrm{Spin}(p,q)$

Ex) If $(V,g) = \mathbb{R}^3$ we have:

$$\begin{array}{c} \mathrm{Spin}(3) \cong SU(2) \\ \downarrow {\scriptstyle 2\text{-}1 \;\&\; \text{onto}} \\ SO(3) \end{array}$$

Last Fall (2002) we worked out this double cover.

Ex) If $(V,g) = \mathbb{R}^{p,q}$

did Fall 2002

$$\begin{array}{c} \mathrm{Spin}_0(3,1) \cong SL(2,\mathbb{C}) \\ \downarrow {\scriptstyle 2\text{-}1 \;\&\; \text{onto}} \\ SO_0(3,1) \end{array}$$

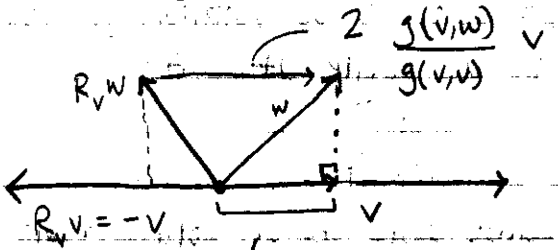
<u>Thm</u>: $O(V,g)$ is generated by <u>reflections along vectors</u>:

given $v \in V$, if $g(v,v) \neq 0$ (length$^2 \neq 0$) then there's a unique element $R_v \in O(V,g)$ st

$$R_v v = -v, \text{ and}$$

if $w \in V$ has $g(v,w) = 0$ then $R_v w = w$.

<u>pf</u>:



Write w as a sum of: part proportional to v, part orthogonal to v.

$$\frac{g(v,w)}{g(v,v)} v$$

We need to divide by length of v squared because as v lengthens, proj v doesn't change!

Check! When v's a unit vector, $\text{proj } v = (v \cdot w)\, v$

So $\boxed{R_v w = w - 2 \frac{g(v,w)}{g(v,v)} v}$

This should remind you of Clifford algs!

This formula says that R_v is unique; you can check

$g(R_v w, R_v w') = g(w,w') \quad \forall\, w,w'$. Check this geometrically or brutally.

Now show every elt of $O(V,g)$ is a product of these reflections.

Now — we consider only g which are positive definite, i.e. wlog we'll use $\mathbb{R}^n$ w/ usual dot product.

R_v acts as id. on vectors $\perp$ to v.

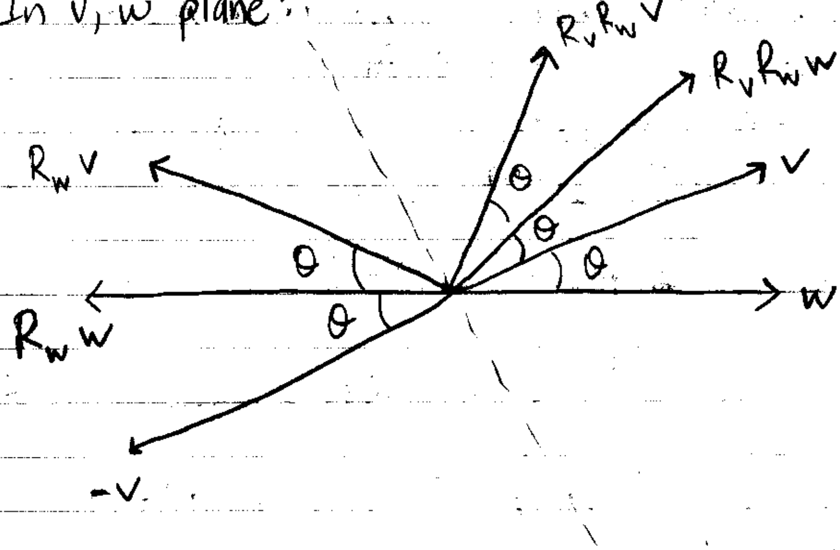
<u>Lemma</u> – Given $v, w \neq 0$ in $\mathbb{R}^n$ if the angle between v and w is θ, then $R_v R_w \in O(n)$ is a <u>rotation</u> <u>in the v, w plane</u> by an angle of 2θ.

i.e. $R_v R_w$ acts as identity on vectors that are $\perp$ to v, w and acts as a rotation on the plane spanned by v, w.

(This is how we get rotations from reflections!)

<u>pf</u> – R_v acts as identity on vectors $\perp$ to v; ditto for w, so $R_v R_w$ acts as id. on vectors $\perp$ to v and w.

In v, w plane:



Note — These R's form a quandle similar to rotation quandle.

So — we see $R_v R_w$ rotates by 2θ "from w towards v."

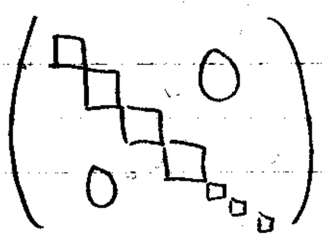
(R_v makes sense when $g(v,v) \neq 0$, so for Minkowski spacetime, can't use "lightlike vectors")

<u>Lemma</u> — Given any elt $h \in O(n)$ we can find some orthonormal basis of $\mathbb{R}^n$ with respect to which its matrix is block diagonal w/ blocks of the form:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } (\pm 1)$$

If V is odd-dim'l, can't use 2×2 matrices!

<u>Block Diagonal</u>:



2×2 blocks &
1×1 blocks

<u>pf</u>: Note $O(n) \subseteq U(n) = \{n \times n \text{ unitary matrices}\}$

Any unitary matrix can be diagonalized and then looks like:

$$\begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} \qquad \theta_i \in \mathbb{R}$$

The eigenvalues $e^{i\theta_j}$ are the roots of the characteristic poly: $\det(\lambda I - h)$

$$\det(\lambda I - h) = (\lambda - e^{i\theta_1}) \cdots (\lambda - e^{i\theta_n})$$

If $h \in O(n)$ this polynomial has <u>real</u> coefficients, so $e^{i\theta_j}$ can either be real (± 1) or come along with its conjugate: $e^{-i\theta_j}$.

(roots of a real poly can be complex, but then the root comes along w/ its complex conjugate)

The first case gives a 1×1 block: (± 1).

The second case gives a 2×2 block:

$$\begin{pmatrix} e^{i\theta_j} & 0 \\ 0 & e^{-i\theta_j} \end{pmatrix}$$

describes spinor rotation about z-axis

Then we can do a change of basis:

$$e_1 \longmapsto e_1 + i e_2$$

$$e_2 \longmapsto e_1 - i e_2$$

to make our 2×2 block become

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We can figure out how many rotations in the plane determine a rotation of space.

Each rotation in plane is product of 2 reflections.

<u>Lemma</u> – Any elt. $h \in SO(n)$ is a product of at most $\lfloor \frac{n}{2} \rfloor$ — greatest integer $\le \frac{n}{2}$.

rotations in planes.

<u>proof</u>: In a suitable basis, h looks like:

$$\begin{pmatrix} \boxed{\begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix}} & & & 0 \\ & \boxed{} & & \\ & & \boxed{} & \\ 0 & & & \boxed{\begin{matrix} 1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{matrix}} \end{pmatrix}$$

det of each block $= 1$,

with an even number of -1 blocks since $\det(h) = 1$

But:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \theta = \pi.$$

So this lets us write h as a product of matrices like:

rotates only these chosen planes, keep rest fixed

$$\begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{pmatrix}$$

This is a rotation in the plane.

Odd # of 1's – <u>NOT</u> a rotation of plane.

When n is odd – have a 1×1 block that's a 1.

Note – we can use exactly $\lfloor \frac{n}{2} \rfloor$ of them.

In 3-d every rotation can be written as 1 rotation of plane.

Ex) In 3-d every rotation is a rotation in a plane:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ in some basis.

<u>Lemma</u> – Any elt $h \in O(n)$ is a product of at most $2\lfloor \frac{n}{2} \rfloor + 1$ reflections along vectors.

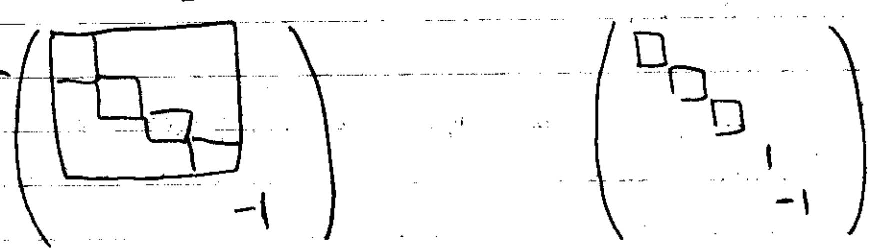
reflections – have $\det = -1$

<u>pf</u> – If $\det(h) = 1$, $h \in SO(n)$ so it's a product of at most $\lfloor \frac{n}{2} \rfloor$ rotations in planes,

each which is a product of 2 reflections, so we need $2\lfloor \frac{n}{2} \rfloor$ reflections.

If $\det(h) = -1$ we can write $h = R_v h'$ with $\det(h') = 1$ since $\det(R_v) = -1$ (only change sign of 1 guy to reflect) So, h is a product of $2\lfloor \frac{n}{2} \rfloor + 1$ reflections. ☐

$2\left(\frac{n-1}{2}\right)+1$ reflections i.e. n



n odd, $\det(h) = -1$

$2\left(\frac{n-2}{2}\right)+1$ = $n-1$ reflections

n even $\det(h) = -1$

Note – conjugation – nice way to get an action of something on something else.

In fact – we only need $\underline{\underline{n}}$ reflections.

Note –

$$V \subseteq C(V,g)$$ — these guys are products of elts. of V

and if $v \in V$ w/ $g(v,v) \neq 0$ then we can think of v as an elt of Clifford alg & it's invertible in $C(V,g)$:

$$v^2 = g(v,v)\,1$$

so, $$v^{-1} = \frac{v}{g(v,v)} \in V \subseteq C(V,g)$$

Let $Pin(V,g)$ be the group inside $C(V,g)$ (not subgrp since $C(V,g)$ isn't a grp) generated by unit vectors, i.e. $v \in V$ w/ $g(v,v) = \pm 1$.

Why??

If $x \in V$, then

Clifford alg conditions:

$$vx + vx = 2g(v,x)\,1$$

so

$$vx = 2g(v,x)\,1 - xv$$

$$C(V,g) \ni vxv^{-1} = \frac{1}{g(v,v)} vxv$$

makes sense in $C(V,g)$

$$= \frac{1}{g(v,v)}\left(2g(v,x)\,1 - xv\right)v$$

$$= \frac{1}{g(v,v)}\left(2g(v,x)\,v - x\,g(v,v)\,1\right)$$

$$= \frac{2g(v,x)}{g(v,v)}\,v - x = -\underset{\substack{\wedge \\ V}}{R_v}\,x$$

see box on prev pgs

Thm – $\mathrm{Pin}(V,g)$ has a representation ρ on V given by:

$$\rho(h)x = hxh^{-1}, \quad h \in \mathrm{Pin}(V,g), \; x \in V$$

and this gives a homo

$$\rho : \mathrm{Pin}(V,g) \longrightarrow O(V,g)$$

(Note – $\mathrm{Pin}(V,g)$ is generated by unit vectors (not everything in $\mathrm{Pin}(V,g)$ is a unit vector) but the calculation we did previously holds in $\mathrm{Pin}(V,g)$.)

pf – Clearly $\rho(hh') = \rho(h)\rho(h')$ and $\rho(1) = 1$

This shows ρ is a rep. (Need these to hold, so that ρ is a rep).

and $\rho(h) : V \longrightarrow V$ since h is in $\mathrm{Pin}(V,g)$, hence a product of unit vectors $h = v_1 \cdots v_m$ and for each unit vector, we've seen $\rho(v_i) : V \longrightarrow V$, $x \longmapsto -R_{v_i} x$

so

$$\rho(h) = \rho(v_1 \cdots v_m) = \rho(v_1) \cdots \rho(v_m).$$

Thus $\rho(h)$ maps V to V since each $\rho(v_i)$ does.

Therefore, ρ is a rep of the group $\mathrm{Pin}(V,g)$ on V.

$\det(\text{reflection}) = -1$

Now we have to check ρ takes us to $O(V,g)$...

Also: We have $\rho(h) \in O(V,g)$ for all $h \in Pin(V,g)$ since $h = v_1 \cdots v_m$ and $\rho(v_i) = -R_{v_i} \in O(V,g)$.

* reflections are orthogonal, so − reflection = orthogonal.

$$\rho(h) = \rho(v_1) \cdots \rho(v_m) \in O(V,g). \blacksquare$$

each are orthogonal, and since $O(V,g)$ is a group, product of orthogonal is orthogonal

<u>Note</u>: If $\dim V$ is odd, $\det(-1_V) = -1$, so

$\det(-R_v) = +1$, so in this case $\det(\rho(h)) = 1$ $\forall\, h \in Pin(V,g)$, so

$$\rho : Pin(V,g) \longrightarrow O(V,g)$$

is <u>NOT</u> onto.

<u>Solution</u>: Define $\tilde{\rho} : Pin(V,g) \longrightarrow O(V,g)$ a homo s.t.

$$\tilde{\rho}(v) = R_v x = (-v x v^{-1}).$$ where v is a unit vector.

May be a problem if $v \in Pin(V,g)$ can be written as a product of both even & odd number of unit vectors. Then do we use $\tilde{\rho}$ or ρ? Do they agree?

Fortunately— this will never happen!

Note — ρ is well defined since we defined it as vxv^{-1}.

In fact — every elt in $\mathrm{Pin}(V,g)$ is either a product of an even number of unit vectors, or odd number, BUT NOT BOTH, so we can define

$$\varepsilon(h) = \begin{cases} +1 & h \text{ a product of an even \# of unit vectors} \\ -1 & h \text{ a product of an odd \# of unit vectors} \end{cases}$$

for $h \in \mathrm{Pin}(V,g)$.

and let

$\tilde{\rho}(h) = \varepsilon(h)\rho(h)$, and note that $\tilde{\rho}$ is a homo:

$$\begin{aligned} \tilde{\rho}(h)\tilde{\rho}(h') &= \varepsilon(h)\rho(h)\varepsilon(h')\rho(h') \\ &= \varepsilon(h)\varepsilon(h')\rho(h)\rho(h') \qquad \varepsilon(h') = \pm 1 \text{ so can pull out front.} \\ &= \varepsilon(hh')\rho(hh') \\ &= \tilde{\rho}(hh') \qquad \text{and } \tilde{\rho}(1) = 1. \end{aligned}$$

Note — The "BUT NOT BOTH" follows from the relns in the Clifford alg since same # of terms on both sides.

guys in $O(V,g)$ are always a product of reflections.

If $\dim V > 0$,

<u>Thm</u> - $\tilde{\rho} : Pin(V,g) \longrightarrow O(V,g)$ is 2-1 & onto.

<u>pf</u> - It's onto since any element of $O(V,g)$ is a product of reflections, and reflections are in the image of ρ since

$$\tilde{\rho}(v) = R_v,$$

where we can assume wlog that v is a unit vector since

$$R_{cv} = R_v \quad \mathbb{R} \ni c \neq 0.$$

follows from formula for reflections.

$\tilde{\rho}$ is at least 2-1 since $1, -1 \in C(V,g)$ both lie in $Pin(V,g)$ and $\tilde{\rho}(1) = \tilde{\rho}(-1) = 1$.

$1 \in Pin(V,g)$ is the identity. Why is -1 in $Pin(V,g)$?

Let v be a unit vector : $g(v,v) = \pm 1$.
If $g(v,v) = -1$, we're done! Since then

$$v^2 = g(v,v)1 = -1$$

is in $Pin(V,g)$ since a grp so closed under mult.

If v is a unit vector, so is $-v$.

If $g(v,v) = +1$, then v & $-v$ are unit vectors, so

$$v(-v) = -g(v,v)1 = -1 \text{ is in } Pin(V,g).$$

$\cap$
$Pin(V,g)$

Finally – note $\tilde{\rho}(1) = 1$ and

$$\tilde{\rho}(-1)x = -1x(-1)^{-1}$$
$$= x$$

so $\tilde{\rho}(-1) = 1$. So $\tilde{\rho}$ is at least 2-1 since it sends 2 guys: 1 & -1 to 1.
To show it's 2-1, we must show
$\ker \tilde{\rho} = \{\pm 1\}$; $\boxtimes$

Finally, $\tilde{\rho}$ restricts to

$$\tilde{\rho} : \mathrm{Spin}(V, g) \longrightarrow SO(V, g)$$

where $\mathrm{Spin}(V, g)$ is generated by elts vv' where v, v' are unit vectors.

4/15/03 Pinors & Spinors.

Given a v. space V w/ metric g on it, we define the Clifford algebra

$$C(V,g) := \text{the alg. generated by } V \text{ modulo the relns } vw + wv = 2g(v,w)1, \quad v,w \in V$$

This is a $\mathbb{Z}_2$-graded algebra.

Defn: Given a group G, we define a G-graded algebra to be an algebra A with a decomposition:

check!

$$A = \bigoplus_{g \in G} A^g \qquad \text{(an alg } \forall \text{ grp elt)}$$

where A^g are subspaces s.t.

$$x \in A^g \text{ and } y \in A^h \Rightarrow xy \in A^{gh}.$$

Elements in A^g are called homogeneous of degree g.

Examples: ① The tensor algebra TV (gen. by v. space V w/ no relns) is $\mathbb{Z}$-graded w/

$$T^n V = \{\text{lin. combs of } v_1 \otimes \ldots \otimes v_n \mid v_i \in V\}$$

(homogeneous tensors of degree n.

$$T^n V \times T^m V = T^{n+m} V$$

mult. is concatenation.

② The symmetric algebra (aka polynomial alg)

$$SV = TV / \langle vw = wv \rangle$$

is also $\mathbb{Z}$-graded. Polys. in formal variables.

$$S^n V = \{ \text{lin. combs of } v_1 \cdots v_n \mid v_i \in V \}$$

Everything in our reln: $vw = wv$ is of degree 2, so doesn't screw up our $\mathbb{Z}$-grading.

③ The exterior algebra ΛV is

$$\Lambda V = TV / \langle vw = -wv \rangle$$

is also $\mathbb{Z}$-graded.
Again our relns are homogeneous — both sides are of degree 2, so doesn't change grading.

$$\Lambda^n V = \{ \text{lin combs. of } v_1 \wedge \ldots \wedge v_n \mid v_i \in V \}$$

But – the Clifford alg. doesn't have homogeneous relations!

$$\underbrace{vw + wv}_{\text{degree } 2} = \underbrace{2g(v,w)\,1}_{\text{degree } 0.}$$

* So we're identifying something of deg. 2 w/ something of degree 0, so this affects our grading!

- mult 2 odds, get an even (because we add the subscripts) we're using multiplicative & additive notation.

④ $C(V,g)$ is only $\mathbb{Z}_2$-graded since the reln:

$vw + wv = 2g(v,w)1$ identifies something of degree 2 w/ something of degree 0.

even part $C^0(V,g) = \{\text{lin. combs of } v_1 \cdots v_n \mid n \text{ even}\}$

odd part $C^1(V,g) = \{\text{lin. combs of } v_1 \cdots v_n \mid n \text{ odd}\}$

Given $x, y \in C^0(V,g)$ then $xy \in C^0(V,g)$, so $C^0(V,g)$ is a subalgebra. The odd part, $C^1(V,g)$ is NOT a subalgebra.

Defn: $Pin(V,g)$ is the group inside $C(V,g)$ generated by unit vectors $v \in V$ i.e. vectors w/ $g(v,v) = \pm 1$.

We can talk about whether an elt of $Pin(V,g)$ is even or odd.

We can define

$$\varepsilon(h) = \begin{cases} 1 & h \in Pin(V,g) \cap C^0(V,g) \\ -1 & h \in Pin(V,g) \cap C^1(V,g) \end{cases}$$

giving a homo

$$\varepsilon : Pin(V,g) \longrightarrow \mathbb{Z}_2$$

We have a 2-1 onto homo:

$$\tilde{\rho} : Pin(V,g) \longrightarrow O(V,g) \quad \text{defined by}$$

$$\tilde{\rho}(h)x = \varepsilon(h)\, hxh^{-1}$$

mult in $C(V,g)$

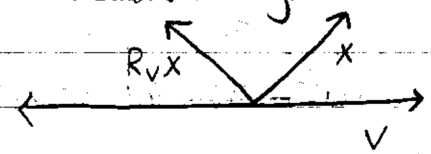
"super conjugation"

- $\mathbb{Z}_2$-graded algebras = "super" algebras
* In these algebras, when we switch 2 things of odd degree, we add on a minus sign.

We saw last time that if v is a unit vector $v \in V$ then

$$\tilde{\rho}(v)x = R_v x$$

where $R_v x$ is x reflected along v:



unit vectors get mapped to reflections.

$$v \longmapsto R_v \qquad -v \longmapsto R_v$$

reflecting along v = reflecting along $-v$.

Shows map $\tilde{\rho}$ is 2-1.

Since $R_v = R_{-v}$; we see that $\tilde{\rho}$ is at least 2-1.

<u>Defn</u>: The <u>spin group</u> is

$$\mathrm{Spin}(V,g) = \mathrm{Pin}(V,g) \cap C^0(V,g)$$

is a subgroup of $\mathrm{Pin}(V,g)$. This is generated by a product of 2 vectors: elts vw where v, w are unit vectors.

spinors are reps of Spin grp, pinors are reps of Pin grp.

<u>Note</u>:

$$\tilde{\rho}(vw) = \tilde{\rho}(v)\tilde{\rho}(w) = R_v R_w$$

= a rotation by angle 2θ in v, w plane where θ is angle bet v & w.

R_v has det -1. $\quad R_v R_w$ has det 1.

Each R_v switches orientation once, so $R_v R_w$ is a rotation in $SO(V,g)$.

Since the elts that generate $Spin(V,g)$ are rotations, then everything in $Spin(V,g)$ is a rotation.

We have $\tilde{\rho}(vw) = R_v R_w \in SO(V,g)$ so we get:

$$\tilde{\rho} : Spin(V,g) \longrightarrow SO(V,g)$$

<u>Pinors & Spinors</u> – representations of Pin & Spin grps.

Many of the reps of these grps come from reps of $SO(V,g)$ & $O(V,g)$.

Pin(V,g) has certain reps coming from reps of $C(V,g)$.

<u>Defn</u>: A <u>representation</u> of an algebra (grp) A is an alg. (grp) homo

$$\rho : A \longrightarrow End(X)$$

where X is v. space and $End(X)$ is the alg. of all linear maps $T: X \to X$.

If we wanted a grp rep: we'd have $End(X)$ = grp of all invertible endos of a v. space.

As w/ group representations, we can define

1) subrepresentations

2) direct sum of representations

3) irreducible representation, or irrep. — one w/out nontrivial subreps
one w/out any subreps other than itself & 0-dim'l rep.

4) indecomposable representation
- one that's not a direct sum of nontrivial reps.

If we have a rep

$$\rho : C(V,g) \longrightarrow End(X)$$

we get a rep of $Pin(V,g)$:

$$Pin(V,g) \hookrightarrow C(V,g) \xrightarrow{\rho} End(X)$$

by composing above. In fact, we get invertible elts of $End(X)$.

What are reps of $C(V,g)$?

We know

$$C(V,g) = \mathbb{K}[n] \; : \; n \times n \text{ matrices where } \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$$

or

$$\mathbb{K}[n] \oplus \mathbb{K}[n] \quad \text{where } \mathbb{K} = \mathbb{R}, \mathbb{H}$$

Note - n is only a power of 2.

Matrices - want to do matrix mult. on vectors, so want to act as linear transf. on the v. spaces $\mathbb{K}^n$.

We say 0-dim'l rep isn't an irrep.

<u>Thm</u> – The only (nontrivial) irrep of $\mathbb{K}[n]$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, is the obvious irrep on $\mathbb{K}^n$.

<u>Thm</u> – The only irreps of $A \oplus B$ where A and B are algebras are those coming from irreps of A or B.

$$A \oplus B \xrightarrow{\pi_1} A \xrightarrow{\rho} \mathrm{End}(X)$$

an irrep of A.

or

$$A \oplus B \xrightarrow{\pi_2} B \xrightarrow{\rho} \mathrm{End}(X)$$

an irrep of B.

(trivial rep is 1-dim'l)

So: we define the <u>pinor reps</u> of $\mathrm{Pin}(V,g)$ to be those coming from irreps of $C(V,g)$; there will either be <u>one</u> irrep $P(V,g)$ if $C(V,g) \cong \mathbb{K}[n]$ or <u>two</u> $P(V,g)$ if $C(V,g) \cong \mathbb{K}[n] \oplus \mathbb{K}[n]$.

What are the pinor reps of

<u>Example</u>: $\mathrm{Pin}(3,1) \subseteq C_{3,1}$ and $\mathrm{Pin}(1,3) \subseteq C_{1,3}$

$C_{3,1} \cong \mathbb{R}[4]$, so $P_{3,1} \cong \mathbb{R}^4$ $\qquad \mathrm{Pin}(3,1) = P_{3,1}$

$\mathbb{R}^{3,1}$

$C_{1,3} \cong \mathbb{H}[2]$, so $P_{1,3} \cong \mathbb{H}^2$

we're just giving the v. space here! Rather than giving the End. (See notes from Fall — did same for grp reps)

Similarly we define <u>spinor reps</u> of $Spin(V,g)$ to be those coming from irreps of $C^\circ(V,g)$. (the even part of Clifford alg).

<u>Thm</u> — $C^\circ_{p,q} \cong C_{p,q-1}$, if $q>0$.

even part of Clifford alg w/ p square roots of 1, q square roots of -1.

<u>pf</u> - We construct a homo

$$\alpha : C_{p,q-1} \longrightarrow C^\circ_{p,q} \quad \text{as follows:}$$

$$\alpha(e_i) = e_i e_n \qquad n = p+q \qquad i = 1, \dots, p+q-1$$

$e_i = \sqrt{1}$ or $\sqrt{-1}$

where e_n is the <u>last square root of -1</u> in $C_{p,q}$.

e_i's generate $C_{p,q-1}$ so we just need to check that $e_i e_n$ satisfy the same relns as e_i.

We 1st list square roots of +1

We'll square both sides & see if we get ± 1.

$$(e_i e_n)^2 = e_i e_n e_i e_n$$
anticommute
$$= -e_i^2 e_n^2$$

$$= e_i^2$$

check they anticommute:

Now check if $i \neq j$

$$(e_i e_n)(e_j e_n) = -e_i e_j e_n^2$$

because e_n anticommutes w/ e_i, e_j.

$$= e_i e_j$$

"Once you're half way across the bridge you can declare you're all the way over by symmetry."

$$(e_i e_n)(e_j e_n) = -e_i e_j e_n^2$$

$$= e_i e_j = -e_j e_i = e_j e_i e_n^2$$

$$= -(e_j e_n)(e_i e_n)$$

[Note — even part of Clifford alg is half as big as whole Clifford alg.]

We could prove α is 1-1 & onto by seeing that it maps the basis

$$e_{i_1} \cdots e_{i_n} \qquad i_1 < \dots < i_n$$

of $C_{p,q-1}$ to some basis of $C^0_{p,q}$. $\blacksquare$

In fact $C^0_{p,q}$ is always either $\mathbb{K}[n]$ or $\mathbb{K}[n] \oplus \mathbb{K}[n]$ so

Spin (p,q) has either one spinor rep, $S_{p,q}$, or two, $S^{\pm}_{p,q}$ which are called left and right-handed spinors.

In fact pinor reps are irreps of the Pin groups and spinor reps are irreps of the Spin grps.

Example: $C^0_{3,1} \cong C_{3,0} \cong \mathbb{C}[2]$ $\longrightarrow$ want to act on $\mathbb{C}^2$.

$$S_{3,1} = \mathbb{C}^2$$

Example: $C^{\circ}_{1,3} \cong C_{1,2} \cong \mathbb{C}[2]$ so $S_{1,3} = \mathbb{C}^2$.

hmwk last week

Note: $C_{p,q+1} = C_{p+1,q}$

Comes from thinking of even Clifford algs as fundamental!

Derek's Thm: $C^{\circ}_{p,q} \cong C^{\circ}_{q,p}$.

pf – $C^{\circ}_{p,q} \cong C_{p,q-1}$ and

$C_{p+1,q} \cong C_{q+1,p}$ ∎

So – $S_{p,q} \cong S_{q,p}$ so for **spinors** the choice of sign convention in g is irrelevant.

* But for **pinors** it **is** relevant.

Now we complexify things.... i.e. $\otimes$ w/ $\mathbb{C}$.

Note: $U(1)$ corresponds to charged particles.
So we want our reps to act on $U(1)$
(be reps of $U(1)$ also).

Complex Pinors & Spinors:

To handle electric charge we like our reps to be complex vector spaces.

So we "complexify" our Clifford algebras:

$$\boxed{\mathbb{C}(V,g) := \mathbb{C} \otimes C(V,g)}$$

Note:

$$\mathrm{Pin}(V,g) \hookrightarrow C(V,g) \hookrightarrow \mathbb{C}(V,g)$$

$C(V,g)$: the real alg. generated by V mod Clifford relns (real lin. combs of things)

$\mathbb{C}(V,g)$: (complex lin. combs of things) complex alg. gen by V mod Clifford relns

and we call the reps of $\mathrm{Pin}(V,g)$ coming from irreps of $\mathbb{C}(V,g)$ complex pinors — either one

$$P^{\mathbb{C}}(V,g), \text{ or two } P^{\mathbb{C}\pm}(V,g).$$

Example:

$\mathbb{C}_{3,1} := \mathbb{C} \otimes C_{3,1} = \mathbb{C}[4]$ so irrep is on $\mathbb{C}^4$

$$P^{\mathbb{C}}_{3,1} \cong \mathbb{C}^4$$

$$\mathbb{C}_{1,3} := \mathbb{C} \otimes C_{1,3} = \mathbb{C} \otimes \mathbb{H}[2] = \mathbb{C}[2][2]$$

$= \mathbb{C}[4]$ so $P^{\mathbb{C}}_{1,3} \cong \mathbb{C}^4$

Also:

$$\mathrm{Spin}(V,g) \hookrightarrow C^\circ(V,g) \hookrightarrow \mathbb{C}^\circ(V,g).$$

and we call the reps of $\mathrm{Spin}(V,g)$ coming from irreps of $\mathbb{C}^\circ(V,g)$ <u>complex</u> <u>spinors</u>: either one

these are actually reps of Pin grp

$S^{\mathbb{C}}(V,g)$ or two $S^{\mathbb{C}\pm}(V,g)$.

$S^{\mathbb{C}}(V,g)$ is called <u>Dirac spinors</u>. These are good for spin-½ particles w/out chirality or handedness and w/ electric charge (get an action of $U(1) \subseteq \mathbb{C}$).

$S^{\mathbb{C}\pm}(V,g)$ are called <u>Weyl</u> <u>spinors</u> - same but w/ handedness.

<u>Example</u>: $\mathbb{C}^\circ_{3,1} = \mathbb{C} \otimes C^\circ_{3,1} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}[2]$

$$\cong (\mathbb{C} \oplus \mathbb{C})[2]$$

$$\cong \mathbb{C}[2] \oplus \mathbb{C}[2].$$

We have 2 irreps, so we have left & right handed Weyl spinors.

$$S^{\mathbb{C}\pm}_{3,1} \cong \mathbb{C}^2.$$

Note - neutrinos are described by guys in here.

$$\mathbb{C}^\circ_{1,3} = \mathbb{C} \otimes C^\circ_{1,3} = \mathbb{C} \otimes \mathbb{C}[2] \cong \mathbb{C}[2] \oplus \mathbb{C}[2]$$

Same as above! $S^{\mathbb{C}\pm}_{1,3} \cong \mathbb{C}^2$. Weyl spinors