4/21/03

Clifford algs. related to normed algebras
(tomorrow - how related to Lie grps, which are the gauge grps)

Normed Division Algebras:

Usually algebra includes associativity but not for today!

Defn: A normed division algebra is a finite-dim'l real vector space $\mathbb{K}$ w/ bilinear multiplication

$$\begin{aligned} m: \mathbb{K} \times \mathbb{K} &\longrightarrow \mathbb{K} \\ (x,y) &\longmapsto xy \end{aligned}$$

unit $0 \neq 1 \in \mathbb{K}$ st $\quad 1x = x1 = x$

and a norm

$$\|\cdot\| : K \longrightarrow [0, \infty)$$

st

$$\|xy\| = \|x\| \, \|y\|$$

Note:

$$\|x\| = \|x1\| = \|1\| \|x\|$$

$$\Downarrow \qquad \text{if } x \neq 0, \text{ eg } x = 1$$

$$\|1\| = 1$$

Also: $\quad xy = 0 \Rightarrow x = 0$ or $y = 0$.

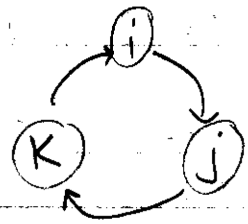
Examples:

① $\mathbb{R}$ w/ usual norm $\|a\| = \sqrt{a^2}$ $\quad$ $\dim \mathbb{R} = 1$

② $\mathbb{C}$ w/ usual norm $\|a+bi\| = \sqrt{a^2+b^2}$ $\quad$ $\dim \mathbb{C} = 2$
(Hamilton discovered that $\mathbb{C}$ consists of just pairs of real numbers w/ a new product)

(there are no 3-dim'l normed div algs)

③ $\mathbb{H}$ w/ usual norm $\|a+bi+cj+dk\| = \sqrt{a^2+b^2+c^2+d^2}$
and $i^2 = j^2 = k^2 = -1$, $ij = k$ & $ji = -k$ & cyclic perm.
quaternions contain dot & cross product (Gibbs)

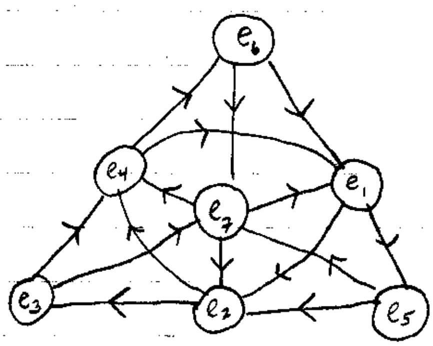


$\dim \mathbb{H} = 4$

④ $\mathbb{O} = \{ \underbrace{a_0 + \sum_{i=1}^{7} a_i e_i}_{a} \mid a_0, \ldots, a_7 \in \mathbb{R} \}$ $\quad$ $\dim \mathbb{O} = 8$

with
$$\|a\| = \sqrt{\sum_{i=0}^{7} a_i^{\,2}}$$
and mult. given by

$e_i^2 = -1$, $i = 1, \ldots, 7$



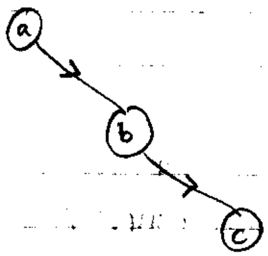
7 pts, 7 lines
(circle is a line)

every line has 3 pts

7 points, 7 cyclically ordered lines

we multiply by — 2 things on same line, its product is the 3rd thing on that line.

ex)



$a \cdot b = c \qquad a \cdot c = -b$

$c \cdot b = -a$

7 points, 7 cyclically ordered lines, each containing 3 pts, determining a copy of $\mathbb{H}$ inside $\mathbb{O}$.

The setup on prev. pg gives $e_1 e_2 = e_4$ and

$$\boxed{e_{n+1} e_{n+2} = e_{n+4 \pmod 7}}$$

And, $e_i e_j = e_k \Rightarrow e_{2i} e_{2j} = e_{2k \pmod 7}$

doubling – corresponds to rotating picture by 1/3.

Check for these examples that $\|ab\| = \|a\|\|b\|$.

<u>Thm</u>: Every normed division alg. is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

We'll prove part of this.

proof: Let $\mathbb{K}$ be a normed division alg.

Given $x \in \mathbb{K}$ let L_x be a linear map:

$$\begin{aligned} L_x : \mathbb{K} &\longrightarrow \mathbb{K} \\ y &\longmapsto xy \end{aligned}$$

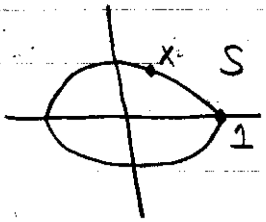
Want to show the norm comes from an inner product.

Prop ①: There's an inner product $\langle \cdot , \cdot \rangle$ on $\mathbb{K}$ s.t. $\|x\| = \sqrt{\langle x, x \rangle}$.

pf — (sketch) Let $S \subseteq \mathbb{K}$ be the unit sphere:

$$S = \{ x \in \mathbb{K} \mid |x| = 1 \}.$$

S is convex & balanced (i.e. $\|x\| = \|-x\|$)



For $x \in S$ we have a linear map $L_x : \mathbb{K} \longrightarrow \mathbb{K}$ and $L_x : S \longrightarrow S$

$(y \in S \Rightarrow \|y\| = 1 \Rightarrow \|xy\| = \|x\|\,\|y\| = 1 \Rightarrow xy \in S)$

and $L_x 1 = x \quad (1x = x)$.

But we have no linear map of plane that sends 1 to x & maps this shape to itself!

The only shape for which this will work is an ellipse!

So using geometry, we conclude S is an ellipsoid so $\|\cdot\|$ comes from some $\langle \cdot , \cdot \rangle$. ☒

<u>Prop ②</u>: $\langle xy, xz \rangle = \langle x, x \rangle \langle y, z \rangle$ and

$\langle yx, zx \rangle = \langle x, x \rangle \langle y, z \rangle$.

pf – We have $\|xy\| = \|x\| \|y\|$

$\Rightarrow \|xy\|^2 = \|x\|^2 \|y\|^2$

$\Rightarrow \langle xy, xy \rangle = \langle x, x \rangle \langle y, y \rangle$

Repeated variable y should become y, z so we <u>polarize</u>:

replace "y" by "$y+z$" do again – replace w/ "$y-z$"

$$\langle x(y+z), x(y+z) \rangle = \langle x, x \rangle \langle y+z, y+z \rangle$$

$$- \quad \langle x(y-z), x(y-z) \rangle = \langle x, x \rangle \langle y-z, y-z \rangle$$ then subtract these!

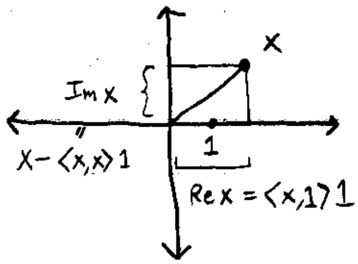
$$2\langle xy, xz \rangle = 2\langle x, x \rangle \langle y, z \rangle$$ ☒

<u>Defn</u>: We define $\boxed{\operatorname{Im} \mathbb{K} = \{ x \in \mathbb{K} \mid \langle x, 1 \rangle = 0 \}}$

So,

$$\mathbb{K} \cong \mathbb{R} \oplus \operatorname{Im} \mathbb{K}$$

$$x \longmapsto (\operatorname{Re} x, \operatorname{Im} x)$$



Note: mult. by an imag # – the product is $\perp$ to what started w/.

where $\operatorname{Re} x = \langle x, 1\rangle 1$
$\operatorname{Im} x = x - \langle x, 1\rangle 1$

so, $x = \operatorname{Re} x + \operatorname{Im} x$.

Prop ③: If $x \in \mathbb{K}$ has $\|x\| = 1$ then $L_x : \mathbb{K} \longrightarrow \mathbb{K}$ is orthogonal,

Check these say same thing!

i.e. $L_x^* L_x = 1$ or

$$\langle L_x y, L_x z\rangle = \langle y, z\rangle$$ (means preserves inner product)

pf –

$$\langle L_x y, L_x z\rangle = \langle xy, xz\rangle = \langle x, x\rangle\langle y, z\rangle$$
$$= \langle y, z\rangle \quad \blacksquare$$

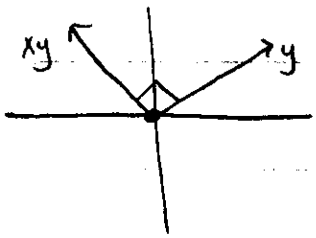
Prop ④: If $x \in \operatorname{Im} \mathbb{K}$ then $L_x : \mathbb{K} \longrightarrow \mathbb{K}$ is skew-adjoint

i.e. $L_x^* + L_x = 0$ or $\langle L_x y, z\rangle + \langle y, L_x z\rangle = 0$ $\forall\, y, z \in \mathbb{K}$.

pf – If $x \in \operatorname{Im} \mathbb{K}$

$$\langle xy, y\rangle = \langle xy, 1y\rangle$$
by Prop 2
$$= \langle x, 1\rangle\langle y, y\rangle$$
since $x \in \operatorname{Im} \mathbb{K}$.
$$= 0$$

Note — $\mathbb{K}$ is a real v. space, so all these are real inner products



How do we get our conclusion? We polarize!

$$\langle x(y+z), y+z \rangle = 0$$
$$- \quad \langle x(y-z), y-z \rangle = 0$$

$$2\langle xy, z\rangle + 2\langle xz, y\rangle = 0$$

$$\Rightarrow \quad \langle L_x y, z\rangle + \langle y, L_x z\rangle = 0$$

get this since inner product is skew-symm.

Prop ⑤: If $x \in \operatorname{Im} \mathbb{K}$ and $\|x\| = 1$, then

$$L_x^2 = -1. \qquad (\text{In particular } x^2 = -1).$$

pf — By previous props ③ & ④ L_x is both orthogonal & skew-adjoint.

orthogonal: preserves inner products

skew-adjoint: maps things to right angles

We know from before that if L_x is orthogonal, you can find some orthonormal basis of $\mathbb{K}$ st it is block diagonal:

$$L_x = \begin{pmatrix} \square & & & & \\ & \square & & 0 & \\ & & \square & & \\ & 0 & & \square & \\ & & & & \square \end{pmatrix}$$

where each block is either

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad (\pm 1)$$

But $L_x^* = -L_x$ (skew adj)

So, we can't have ± 1 blocks, and we need $\cos\theta = 0$.
i.e. we can only have blocks like

$$\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{which is } i.$$

Thus, $L_x^2 = -1$. $\blacksquare$

Or

pf — orthogonal $L_x^* L_x = 1$

skew-adj $L_x^* = -L_x$

$$\Downarrow$$

$$(-L_x)(L_x) = 1$$

$$\Downarrow$$

$$L_x^2 = -1 \quad \blacksquare \checkmark$$

Note - the corresponding thing to do in analysis to polarization is integrate by parts!

Prop ⑥ - If $x \in \operatorname{Im} \mathbb{K}$ then $L_x^2 = -\langle x,x\rangle 1$

pf - $\frac{x}{\|x\|}$ is covered by the prev. prop

$$L^2_{\frac{x}{\|x\|}} = -1 \qquad \frac{1}{\|x\|} \in \mathbb{R}$$

$$\frac{1}{\|x\|} L_x^2 = -1$$

$L_x^2 = -\langle x,x\rangle 1$ unless $x=0$, in which case $L_x = 0$, so $L_x^2 = 0$. ∎

But the above reminds us of the Clifford alg. relation!

Prop ⑦ - If $x,y \in \operatorname{Im} \mathbb{K}$ then $L_x L_y + L_y L_x = -2\langle x,y\rangle 1$ which is just the Clifford alg. relns w/ $g(x,y) = -\langle x,y\rangle$.

pf - Polarize!

$$(L_x + L_y)^2 = L_{(x+y)}^2 = -\langle x+y, x+y\rangle 1$$

$$-\quad (L_x - L_y)^2 = L_{(x-y)}^2 = -\langle x-y, x-y\rangle 1$$

$$L_x L_y + L_y L_x = -2\langle x,y\rangle 1 \quad ∎$$

* Whenever you have something defined in terms of generators &, relns, you can define it as a univ. property! *

ex) free grps, ten. prod. of v. spaces

<u>Prop ⑧</u>: If $\mathbb{K}$ is a normed division alg. then

$$\begin{aligned} L : \operatorname{Im}\mathbb{K} &\longrightarrow \operatorname{End}(\mathbb{K}) \\ x &\longmapsto L_x \end{aligned}$$

extends to a rep of the Clifford alg

$$C(\operatorname{Im}\mathbb{K}, -\langle\cdot,\cdot\rangle) \quad \text{on } \mathbb{K},$$

$$\alpha : C(\operatorname{Im}\mathbb{K}, -\langle\cdot,\cdot\rangle) \longrightarrow \operatorname{End}(\mathbb{K}).$$

pf – $C(V,g)$ is generated by V w/ relns

$$vw + wv = 2g(v,w)$$

i.e. (in terms of a univ. property)

given any map $f : V \longrightarrow A$ where A is an assoc. alg st

$$f(v)f(w) + f(w)f(v) = -2g(v,w)\,1_A$$

there exists a unique homo

$$\alpha : C(V,g) \longrightarrow A \quad \text{st}$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, hook] & A \\ C(V,g) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutes.

Here we get:

$$\begin{tikzcd} \mathrm{Im}\,\mathbb{K} \arrow[r] \arrow[d, hook] & \mathrm{End}(\mathbb{K}) \\ C(\mathrm{Im}\,\mathbb{K}, -\langle\cdot,\cdot\rangle) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

since by Prop 7: $\quad L_x L_y + L_y L_x = -2g(v,w)\,1$. ☒

<u>Prop ⓐ</u>: Any way of making $\mathbb{R}^{n+1}$ into a normed div. alg. gives a rep of $C_{0,n}$ on $\mathbb{R}^{n+1}$ where $C_{0,n}$ is the Clifford alg. generated by n anticommuting square roots of -1.

pf — $\mathbb{R}^{n+1} \cong \mathbb{K}$
$\mathbb{R}^n \cong \mathrm{Im}\,\mathbb{K}$ in prev. prop. ☒

Supposing we know this... when is there an $n+1$ dim'l rep on $C_{0,n}$?

can have action of $\mathbb{H}$ on itself on left & right

<u>n</u>	<u>$C_{0,n}$</u>	smallest rep $P_{0,n}$ (or $P^{\pm}_{0,n}$)	Does $C_{0,n}$ have a rep on $\mathbb{R}^{n+1}$?
0	$\mathbb{R}$	$\mathbb{R}$	Yes — gives $\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{C}$ 2-dim'l	Yes — gives $\mathbb{C}$
2	$\mathbb{H}$	$\mathbb{H}$ 4-dim'l need $n+1=3$	No
3	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}, \mathbb{H}$	Yes — gives $\mathbb{H}$
4	$\mathbb{H}[2]$	$\mathbb{H}^2$	No
5	$\mathbb{C}[4]$	$\mathbb{C}^4$ — this is 8-dim'l need $6=n+1$	No
6	$\mathbb{R}[8]$	$\mathbb{R}^8$	No
7	$\mathbb{R}[8]\oplus\mathbb{R}[8]$	$\mathbb{R}^8, \mathbb{R}^8$	Yes — gives $\mathbb{O}$
8	$\mathbb{R}[16]$	$\mathbb{R}^{16}$ this is 16-dim'l, want $8+1=9$	No (w/ left/right action)
9	$\mathbb{C}[16]$	$\mathbb{C}^{16}$	No
10	$\mathbb{H}[16]$	$\mathbb{H}^{16}$	No

After this, $\dim P_{0,n} > n+1$ by Bott periodicity.

<u>Note</u>: $\mathbb{O}$ isn't assoc, so doing a left mult. then another left mult. <u>isn't</u> the same as a single left mult.

For $\mathbb{O}$, $L_x L_y \neq L_{xy}$

and, in fact, all of $\mathrm{End}(\mathbb{O}) = \mathbb{R}[8]$ is generated by L_x's.

<u>Thm</u>: All normed division algs. have dimension 1, 2, 4 or 8.

(Note — we haven't proved that all normed div algs are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, <u>or</u> the converse of above statement.)

What's left?

Once we know the reps of $C(\overset{\mathrm{Im}\,\mathbb{K}}{\overset{\shortparallel}{\mathbb{R}^n}}, -\langle\cdot,\cdot\rangle)$ on $\underset{\mathbb{K}}{\underset{\shortparallel}{\mathbb{R}\oplus\mathbb{R}^n}}$, we know how to left multiply any elt. of $\mathbb{K}$ by any <u>imaginary</u> elt. of $\mathbb{K}$.

We also know how to left multiply by <u>real</u> elts (just scalar mult by real numbers)

So, $m: \mathbb{K}\times\mathbb{K} \longrightarrow \mathbb{K}$ is determined.

So we have at most

1	1-dim'l	normed division algebras.
1	2-dim'l	
2	4-dim'l	
2	8-dim'l	

Now we need to check: $\|xy\| = \|x\|\|y\|$. But it always holds!

But in 4-dim'l & 8-dim'l cases we get

$$\mathbb{H},\ \mathbb{H}^{op},\ \mathbb{O},\ \mathbb{O}^{op}$$

(where $m(x,y) = yx$ i.e. $xy_{\mathbb{K}^{op}} = yx_{\mathbb{K}}$.

But

$$\begin{array}{ccc} \mathbb{H} \cong \mathbb{H}^{op} & \mathbb{O} \cong \mathbb{O}^{op} & \\ x \longmapsto \bar{x} & x \longmapsto \bar{x} & (\overline{xy} = \bar{y}\bar{x}). \end{array}$$

4/22/03 <u>Compact</u> <u>Lie</u> <u>groups</u>

- in physics — the field eqns that describe laws of nature:
  <u>Yang-Mills eqns</u> — must choose a Lie grp
  (need connections for //-transport)

In physics, we need a compact Lie group to get positive energy in Yang-Mills theory, which describes forces other than gravity. For the <u>Standard</u> <u>Model</u>:

$$G = \underbrace{SU(3)}_{\text{strong}} \times \underbrace{SU(2) \times U(1)}_{\text{electroweak}}$$

is our Lie grp.

In grand unified theories we seek bigger, prettier compact Lie groups.

We can classify compact connected Lie groups!

<u>Note</u>: A finite group is a compact Lie grp!
A finite grp is a collection of points, which is a 0-dim'l manifold.
So we don't include these in our classification (add in "connected" requirement)

We'll approach this via Clifford algs, esp.
$\mathbb{R}, \mathbb{C}, \mathbb{H}$ &, $\mathbb{O}$
not a Clifford alg, but a normed divi. alg.

Suppose $\mathbb{K}$ is a normed division algebra $(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$
Then,

$$\begin{array}{ccccc} x & = & \operatorname{Re} x & + & \operatorname{Im} x \\ & & \cap & & \cap \\ & & \mathbb{R} & & \operatorname{Im} \mathbb{R} \end{array} \qquad \text{For } x \in \mathbb{K}$$

so we can define $\quad \bar{x} = \operatorname{Re} x - \operatorname{Im} x$

and this makes $\mathbb{K}$ into a (not necess. assoc) <u>$*$-algebra</u>.

(Ex. A matrix alg. becomes a $*$-alg w/ transpose of matrices)
A $*$-alg encodes complex conjug.

<u>Defn</u>: A <u>$*$-algebra</u> is an algebra A w/ a linear map

$$* : A \longrightarrow A$$

which is an <u>antihomomorphism</u>:

$$(ab)^* = b^* a^*, \qquad 1^* = 1$$

which is <u>involutory</u>:

$$a^{**} = a.$$

Ex) $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are $*$-algs.

Ex) $M_n(\mathbb{K}) = \mathbb{K}[n]$ is a $*$-alg w/

$$(a_{ij})^* = (\overline{a_{ji}}) \qquad \text{conjugate in } \mathbb{K}.$$

"adjoint of a"

Given a $*$-algebra A, we can define:

$GL(A) = \{a \in A \mid a$ has an inverse a^{-1} s.t. $aa^{-1} = a^{-1}a = 1\}$

$\cup\!\!|$

$U(A) = \{a \in A \mid a^*a = aa^* = 1\}$ (unitary)

$\mathfrak{u}(A) = \{a \in A \mid a^* = -a\}$ (skew adjoint)

Thm – If A is a finite dim'l $*$-alg, then $GL(A)$ is a Lie group w/ Lie alg. $\mathfrak{gl}(A) = A$.
Also $U(A)$ is a Lie group w/ Lie algebra $\mathfrak{u}(A)$.

From here onwards, $\mathbb{K}$, A are associative.

pf – $GL(A)$ is a group (all invertible elts of something is always a grp). $U(A)$ is a subgroup since

$$a, b \in U(A) \Rightarrow (ab)^*(ab) = b^*a^*ab = b^*1b = 1$$

and $(ab)(ab)^* = 1 \Rightarrow ab \in U(A)$.

And, $a \in U(A) \Rightarrow a^{-1} = a^* \in U(A)$ since
$a^*a^{**} = a^{**}a^* = 1$

But why are they Lie grps?

Note: we've only got finite dim'l v.space, so $*$, all maps are cont.

$GL(A)$ is a manifold since it's an open subset of the vector space A. (Here, however, we don't have a notion of det... so can't take the preimage)

<u>Exercise</u> – A invertible, B close to A $\Rightarrow$ B invertible (geometric series)

<u>Closed subgrp Thm</u>: If G is a Lie grp and $H \subseteq G$ is a subgrp that's closed (topologically) then H is a Lie group.

$U(A)$ is a closed subgroup of $GL(A)$ since:

$$a_i \text{ unitary and } a_i \longrightarrow a \Rightarrow a \text{ unitary}$$

since mult, $^{-1}$, ~~*~~ are continuous.

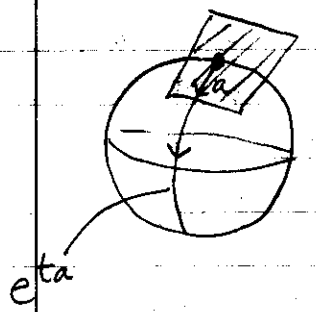
To show the Lie alg. of $GL(A)$ is $\mathfrak{gl}(A) = A$ amounts to showing

$$a \in A \Rightarrow e^{ta} = \sum_{n=0}^{\infty} \frac{(ta)^n}{n!} \in GL(A) \qquad t \in \mathbb{R}$$

e^{ta} is invertible — its inverse is e^{-ta}

Also need: If $f: \mathbb{R} \longrightarrow GL(A)$ is any smooth homomorphism then
$$f(t) = e^{ta} \text{ for some } a.$$

These 1-parameter subgrps correspond to tang. vectors in $\mathfrak{g}$. (1-1 correspondence)



To show $U(A)$ has $\mathfrak{u}(A)$ as its Lie algebra we need to check:

$$a^* = -a \quad \text{iff} \quad e^{ta} \text{ is unitary } \forall\, t \in \mathbb{R}$$

$$\text{i.e. } (e^{ta})^* e^{ta} = e^{ta}(e^{ta})^* = 1$$

$*$ is linear, so goes into sum e^{ta}

$(\Leftarrow)$ $$\frac{d}{dt}(e^{ta})^* e^{ta} = \frac{d}{dt}(1) = 0$$

$$\parallel$$

$$a^* e^{ta^*} e^{ta} + e^{ta^*} a\, e^{ta}$$ now set $t = 0$.

$$a^* + a = 0 \quad \Rightarrow \quad a^* = -a.$$

$(\Rightarrow)$ Use fact $(e^{ta})^* = e^{ta^*}$.

Then $a^* = -a \Rightarrow (e^{ta})^* = e^{-ta}$, so

$$(e^{ta})^*(e^{ta}) = (e^{-ta})(e^{ta}) = e^0 = 1.$$

references:

- Helgason – Differential Geometry, Lie Groups and Symmetries
- J. F. Adams – Lectures on Lie groups

Now let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and let $A = \mathbb{K}[n] = M_n(\mathbb{K})$.
Then $U(A)$ has other names:

- $O(n) = U(\mathbb{R}[n]) = U(M_n(\mathbb{R}))$ the orthogonal grp
- $U(n) = U(\mathbb{C}[n]) = U(M_n(\mathbb{C}))$ the unitary grp
- $Sp(n) = U(\mathbb{H}[n]) = U(M_n(\mathbb{H}))$ the symplectic grp

is connected

or "$Sp(2n)$"

also is a name of another Lie grp.

Their Lie algebras are called:

$$\mathfrak{o}(n),\ \mathfrak{u}(n),\ \mathfrak{sp}(n).$$

The above grps are all about "rotations/reflections" in $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$.

$O(n)$ isn't connected (2 pieces – $\det = 1$, $\det = -1$).

If $\mathbb{K}$ is commutative ($\mathbb{R}, \mathbb{C}$) we can define

$$\det : \mathbb{K}[n] \longrightarrow \mathbb{K}$$

using the usual formula and it satisfies

$$\det(ab) = \det(a)\det(b)$$

No good notion of quaternionic determinant.

$$\boxed{\det(e^{ta}) = e^{t\,tr(a)}}$$ so $tr(a)=0$
$\Downarrow$
$\det(e^{ta})=1.$

Then — we can define

$SO(n) = \{a \in O(n) \mid \det a = 1\}$ and

$SU(n) = \{a \in U(n) \mid \det a = 1\}$

These are subgrps of $O(n)$ & $U(n)$ by det. condition.

Their Lie algebras are:

$so(n) = \{a \in \sigma(n) \mid tr(a) = 0\}$ skew-adjoint

$su(n) = \{a \in u(n) \mid tr(a) = 0\}$ but, $\underline{so(n) = \sigma(n)}$.

check
Above fact — diagonalizable matrices are dense in all matrices, and for diagonal matrices,

$$e\begin{pmatrix} \diagdown \end{pmatrix} = \begin{pmatrix} e^{\circ} & & \\ & e^{\circ} & \\ & & \ddots \\ & & & e^{\circ} \end{pmatrix}$$

so $\det\begin{pmatrix} & \end{pmatrix} = e^{\circ}e^{\circ}e^{\circ} = e^{\text{add up}}$

<u>Prop</u> — $O(n)$, $U(n)$ and $Sp(n)$ are compact Lie grps.

pf — We've seen that they're Lie groups, since $U(\mathbb{K}[n])$ is a Lie grp by prev. prop.

Why compact?

Why compact?

We saw $U(\mathbb{K}[n])$ is a closed subset of $\mathbb{K}[n]$.
Why is it bounded?

Any $a \in \mathbb{K}[n]$ defines an $\mathbb{R}$-linear map from $\mathbb{K}^n$ to itself by matrix mult.

$$(av)_i = \sum_j a_{ij} v_j \qquad v \in \mathbb{K}^n,\ a \in \mathbb{K}[n]$$

So we can define

$$\|a\| = \sup_{v \neq 0} \frac{\|av\|}{\|v\|} \quad \text{where } \|v\|^2 = \sum_i \bar{v}_i v_i$$

Note – w/r/t this norm, $U(\mathbb{K}[n])$ is bounded since a unitary $\Rightarrow \|av\| = \|v\|$.

So, $\|a\| = 1$. $\boxtimes$

Instead of describing what all compact Lie grps are like, we'll describe what the Lie algs. of all compact Lie grps are like.

check – What's true about G, G' if have same Lie alg.

All connected Lie groups w/ same Lie alg. are quotients of some given Lie group G by a $\underbrace{\text{discrete subgrp}}_{\text{0-dim'l Lie grp}}$ normal N.

(finite in compact case).

and conversely!

Note — if G isn't connected, $\mathfrak{g}$ doesn't tell us that!

Examples:

① $O(n)$ & $SO(n)$ have the same Lie alg, but $SO(n)$ is connected and $O(n) \supseteq SO(n)$ has 2 connected components.

② $SO(2n)$ has $\{\pm I\}$ as a normal subgrp.
iden. matrix

$\{\pm I\}$ is the center of $SO(2n)$, so is normal.

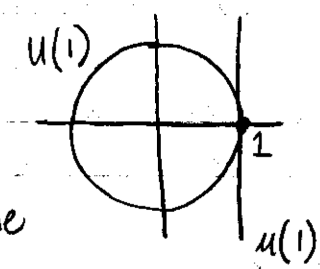
So — $SO(2n)/\{\pm I\}$ has same Lie alg.

as $O(2n)$.

Thm — If G, G' are compact Lie grps, so is $G \times G'$ which has Lie alg. $\mathfrak{g} \oplus \mathfrak{g}'$.

Thm — If G is a compact Lie group, its Lie alg. is a finite direct sum of copies of:

1) $\mathfrak{u}(1)$ the Lie alg. to $U(1)$



$\mathfrak{u}(1) = \{ix \mid x \in \mathbb{R}\}$ the imaginary line
$\cong \mathbb{R}$

This is the 1-dim'l abelian Lie alg: $[x,y] = 0$.

2) $\mathfrak{so}(n)$ (reals)

3) $\mathfrak{su}(n)$ (complex)

4) $\mathfrak{sp}(n)$ (quatenionic version)

5) g_2] related to octonians
↳ not assoc!

6) f_4, e_6, e_7, e_8]

Note: There are some redundancies, e.g.

$$\mathfrak{su}(2) \cong \mathfrak{sp}(1)$$

which comes from Lie grp. fact: $SU(2) \cong Sp(1)$.

$SU(2) =$ 2x2 unitary matrices w/ det = 1

3-sphere $= \{ a + ib\sigma_1 + ic\sigma_2 + id\sigma_3 \mid a^2+b^2+c^2+d^2=1 \}$

$= \left\{ \begin{pmatrix} a+id & ib-c \\ ib+c & a-id \end{pmatrix} \;\middle|\; a^2+b^2+c^2+d^2=1,\ a,b,c,d \in \mathbb{R} \right\}$

$\cong \{ a+bi+cj+dk \mid a^2+b^2+c^2+d^2=1 \}$ quaterions

$= Sp(1) \subseteq \mathbb{H}$

↑ unit quaternions (quat. w/ length 1)

The remaining redundancies involve so's (hmwk).

<u>Note:</u> g_2 is the Lie alg. of Lie grp G_2

where $G_2 = \mathrm{Aut}(\mathbb{O})$ - all alg. autos of $\mathbb{O}$

autos of $\mathbb{O}$ which preserve the product

$\dim G_2 = 14$.

<u>Note:</u> f_4, e_6, e_7, e_8 contain as sub-Lie-algs

$so(9)$, $so(10)$, $so(12)$ and $so(16)$, respectively.

$9 = 8+1$
$10 = 8+2$
$12 = 8+4$
$16 = 8+8$

Bott periodicity says $so(n) \approx so(n+8)$ (close)

1, 2, 4, 8 are dimensions of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Now, let $C_n = C_{0,n}$.

$\boxed{\mathbb{R}^{16} = \mathbb{O}^2}$

n	$C_n = C_{0,n}$	C_n°	Division algs (spinors that are ↑)	
0	$\mathbb{R}$			
1	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	
2	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{C}$	
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$		
4	$\mathbb{H}[2]$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$	
5	$\mathbb{C}[4]$	$\mathbb{H}[2]$		
6	$\mathbb{R}[8]$	$\mathbb{C}[4]$		
7	$\mathbb{R}[8] \oplus \mathbb{R}[8]$	$\mathbb{R}[8]$		
8	$\mathbb{R}[16]$	$\mathbb{R}[8] \oplus \mathbb{R}[8]$	$\mathbb{O}$	
* 9		$\mathbb{R}[16]$	$\mathbb{O}^2$	pairs of octonians
* 10		$\mathbb{C}[16]$	$(\mathbb{O} \otimes \mathbb{C})^2$	bioctonians
11				
* 12		$\mathbb{H}[16] \oplus \mathbb{H}[16]$	$(\mathbb{O} \otimes \mathbb{H})^2$	quateroctonians
13				
14				
15				
* 16		$\mathbb{R}[128]$	$(\mathbb{O} \otimes \mathbb{O})^2$	octo-octonians

* These are the strange dimensions where the exceptional Lie algs live!

f_4, e_6, e_7, e_8 are Lie algs of certain symmetry groups related to

$\mathbb{O}, \mathbb{O} \otimes \mathbb{C}, \mathbb{O} \otimes \mathbb{H}, \mathbb{O} \otimes \mathbb{O}.$

(no rotations in $\mathbb{O}$)

$$f_4 \subseteq e_6 \subseteq e_7 \subseteq e_8$$