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Bott periodicity for complex Clifford algebras

$\mathbb{C}_{p,q}$ = alg. over $\mathbb{C}$ gen. by
- p square roots of 1,
- q square roots of -1, all anticommuting

<u>Thm</u>:

$$\mathbb{C}_{p,q} \cong \begin{cases} \mathbb{C}[n] & p+q \text{ even} \\ \mathbb{C}[n] \oplus \mathbb{C}[n] & p+q \text{ odd} \end{cases}$$

where n is chosen to make dimensions work

$$n = \begin{cases} 2^{p+q/2}, & p+q \text{ even} \\ 2^{p+q-1/2}, & p+q \text{ odd} \end{cases}$$

So —

1) $\mathbb{C}_{p,q}$ only depends on $p+q$

(because we can change sqr. roots of 1 to sqr. root of -1 by mult. by i)

if x is a sqr. root of 1, then ix is a sqr. root of -1.

2) $\mathbb{C}_{p+2,q} \cong \mathbb{C}_{p+1,q+1} \cong \mathbb{C}_{p,q+2} \cong \mathbb{C}_{p,q} \otimes_{\mathbb{C}} \mathbb{C}[2]$

<u>Quaternionic</u> <u>Hilbert</u> <u>spaces</u> (Toby)

R, S, T rings — inside center is a copy of K

We'll define an algebra over R, S, T, but these aren't commutative! (for example, R, S, T could be $\mathbb{H}$).

M is an R-S <u>bimodule</u>: $(\underbrace{M}_{\text{abel. grp}}, +, \underbrace{\text{left mult by } R \text{ scalars}}_{R\times M \to M}, \underbrace{\text{right mult by } S \text{ scalars}}_{M\times S \to M})$

So, M is a left module over R, right module over S but what makes it a <u>bimodule</u> is this compatibility relation:

$$\boxed{(rm)s = r(ms)}$$

<u>Note</u>: There <u>are</u> things which are left modules over R, right modules over S, but <u>NOT</u> R-S bimodules.

Recall— R, S are algs. over K! So, we add in the law

$$km = mk \qquad k \in K,\ m \in M$$

(not part of defn. of bimodule)

<u>Defn</u>: M is an <u>R-S bimodule over K</u> if in addition to being an R-S bimodule, we have

$$\boxed{km = mk}$$

The R & S structures on M makes M into a K-module in 2 ways, so we want this to be the same!

<u>Example</u>: If $f: R \longrightarrow S$ is a ring homo, then S becomes an R-S bimodule by

$$rs := f(r)s$$

and right mult is same as right mult by S.

S becomes an R-S bimodule <u>over</u> <u>K</u> if f is an alg. homo (i.e. preserves mult. by K).

<u>Defn</u>: An <u>R-algebra over K</u> is a K-alg A w/ a K-alg homo $R \longrightarrow A$.

(we want elts of K to commut w/ A, but we <u>don't</u> want R to. R need not be commut.)

or:
- A is an R-bimodule over K. (can mult on left & right by elts in R)
- A is a K-alg

Ex) So many of our Clifford algs become algs over $\mathbb{H}$, etc.

Recall: A bimodule homo $f: M \longrightarrow N$ (R-S bimodules over K)

f is additive
f preserves both scalar mult. $f(rm) = rf(m)$ (*)
$f(ms) = f(m)s$

The set of bimodule homos. is boring!

(*) is bad because we get f commuting w/ r.

(Want the set of all these homos to be a bimodule, but it isn't).

Now, let $f: M \longrightarrow N$ where M is an R-T bimodule over K and N is a S-T bimodule over K.

(4) $f(rm) = (fr)m$

r acts on left, but function is on the left $(fm)t$, So r gets squished in between.

Ex) $L_\ell(\mathbb{R}^n) \cong M_n(\mathbb{R})$ matrix mult. w/ matrix on left.

Ex) $L_\ell(\mathbb{H}^2) \cong \mathbb{H}[2]$

So (5)

$$m(ft) = (mf)t$$
$$m(sf) := (ms)f$$
$$(rm)f = r(mf)$$

Dualss:

${}^{\vee\vee}M$ is an R-S bimodule, but $M \not\to {}^{\vee\vee}M$, $M \not\to M^{\vee\vee}$

but we do get: $M \longrightarrow ({}^{\vee}M)^{\vee}$
$M \longrightarrow {}^{\vee}(M^{\vee})$

Note: ${}^{\vee}M$, $M^{\vee}$ are both S-R bimodules over K.

$$L_{\ell}(S, M) \cong M \cong L_{r}(R, M)$$

Involutions:

Defn: An involution $-$ on R (a K-alg) is a K-alg antiautomorphism w/ $\bar{\bar{\ }} =$ equal to the identity.

$$(\overline{rr'}) = \bar{r}'\bar{r} \quad \text{(antiauto)}$$

There is something like complex conj. on quaternions, and it does this.

Note: $\bar{K} = K$ and K being commutative are different. i.e. $\mathbb{C}$ is commut, but $\bar{z} \neq z \ \forall z \in \mathbb{C}$.

Now we want to put an involution on our bimodules.

Suppose M is an R-bimodule over K (it's necessary that we've got R on both sides!)

$*: M \longrightarrow M$ a K-module iso st $** = \text{id}$.

Want $*$ to be an involution. R is equipped anti-auto: reverses order of mult.

w/ its own involution (doesn't make sense to apply $*$ to things in R). So we have:

$$(rmr')^* = \bar{r}' m^* \bar{r}$$ ($*$ is involution, so should get anti auto — switching)

We call M an $R*$ bimodule over K.

ex) If $r' = 1$, we get $(rm)^* = m^* \bar{r}$

(if we knew how to mult on right, but not left, take $*$ of both sides).

$$rm := (m^* \bar{r})^*$$

So for $R*$ bimodules, we get mult on left & right related.

Defn: If A is an R-alg over K then $*: A \to A$ is a <u>K-alg anti-automorphism</u> w/ $** = id$.

<u>pf of Thm 3:</u>

If $f \in L_\ell(M)$ (action on left) — involution reverses order and is its own inverse.

Define

$$mf := (f^* m^*)^*$$

$\mathbb{R}^n$ has a $*$-structure by taking conjugates componentwise.
$M_n(\mathbb{R})$ has a $*$-structure by taking conj. transpose.

(<u>Note</u>: inner product defines how we get transpose)

$\langle rm, m' \rangle = \langle m, \bar{r} m' \rangle$ since

$$\langle rm, m' \rangle = (rm)^t m' = (m^t \bar{r}) m' = m^t (\bar{r} m')$$
$$= \langle m, \bar{r} m' \rangle$$

Thm 4: $\langle m, m' \rangle_r$

$$\| $$
$$m m'^t = (m m'^t)^{**} = (m'^{t*} m^*)^* = (m'^{*t} m^*)^*$$
$$= \langle m', m \rangle_e^*$$
$$= \overline{\langle m', m \rangle_e} = \langle m, m' \rangle_e$$

Ex) $K = \mathbb{R}$, $R, S = \mathbb{R}$, $M = \mathbb{R}^4$ w/ Minkowski space, we get all on pg w/ Thm 4.

Defn: C^*-alg is a $*$-alg w/ a norm that makes the space into a Banach space and
$|r^* r| = |r|^2$.
$|rr'| \leq |r||r'|$ (equal in dormed div. alg)

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Quantum Theory:

To each physical system we associate a complex Hilbert space. The states of the system (way things can be) are equivalence classes of unit vectors ψ in the Hilbert space H of that system. Processes are linear operators $T: H \longrightarrow H'$ between Hilbert spaces.

Given a state ψ of the system described by H and applying the process $T: H \longrightarrow H'$, what's the probability that it's found in the state $\psi' \in H'$?

The amplitude (not probability) is

$$\langle \psi', T\psi \rangle \in \mathbb{C}$$

(can't be a probability since a complex #).

Note: no-one really knows what the amplitude really is.

If T is unitary, we can get the probability:

$$0 \leq |\langle \psi', T\psi \rangle|^2 \leq 1$$

(since T is unitary, $\|T\psi\| = \|\psi'\| = 1$

(If we let ψ' range over an o.n. basis, we get numbers that add up to 1.)

i.e. $\sum |\langle e_i', T\psi \rangle|^2 = 1$ if e_i' is an orthonormal basis of H'.

(probabilities sum to 1)

says prob. of finding system in state $T\psi$ is also in e_i'

We've never used the fact that our Hilbert spaces are <u>complex</u>, so all of prev. page works for real, complex & quaternionic Hilbert spaces. For $\mathbb{H}$ we need to carefully define this notion — a quaternionic vector space should not be just a left $\mathbb{H}$-module, it needs to be an <u>$\mathbb{H}$-bimodule</u> for left or right endomorphisms to be an $\mathbb{H}$-module.

In fact — it will be an "$\mathbb{H}$-algebra over $\mathbb{R}$."

<u>Symmetries</u>

If a group G acts as <u>symmetries</u> then we want all our Hilbert spaces H to be equipped w/ unitary reps

$$\rho : G \longrightarrow U(H)$$

unitary transf. of H.

in the real case, we call this $O(H)$, quaternionic, $Sp(H)$.

Now we want the operators T to get along w/ ρ.

* [when you add more structure to objects in a category, you want the morphisms to get along w/ and preserve this new structure.]

Similarly, our processes $T : H \to H'$ should all be <u>intertwiners</u>:

$$\boxed{T\rho(g) = \rho'(g)T}$$

often - we want G to be a Lie group. If G is a Lie group, we want the reps ρ to be smooth so we get Lie alg. reps:

$$d\rho : \mathfrak{g} \longrightarrow \mathfrak{u}(H)$$

skew-adjoint operators on H.

Perturbative Quantum Field Theory: (QFT)

In QFT, we assume our systems live in Minkowski space, $\mathbb{R}^{n,1}$ (n post. signs, 1 neg. sign)

$\mathbb{R}^{n,1}$ means $\mathbb{R}^{n+1}$ w/ metric $g(v,w) = \underbrace{v_1 w_1 + \dots + v_n w_n}_{\text{space}} - \underbrace{v_{n+1} w_{n+1}}_{\text{time}}$

$v, w \in \mathbb{R}^{n,1}$.

(want to see how physics looks different in different dimensions.)

Our symmetry group (which will include symmetries of our space $\mathbb{R}^{n,1}$) will therefore include the Poincaré group, which is the symmetry grp of $\mathbb{R}^{n,1}$.

But - we don't want these to be linear transf since these fix an origin, and spacetime doesn't have a fixed origin.

Poincaré group: all smooth maps $f : \mathbb{R}^{n,1} \longrightarrow \mathbb{R}^{n,1}$

preserving "distances" as measured by g:

$$g(f(x)-f(y),\ f(x)-f(y)) = g(x-y,\ x-y)$$

Note – transformations aren't linear $f(x+y) \neq f(x) + f(y)$.

This group, also called $IO(n,1)$ "inhomogeneous"

Toby – proved this for Euclidean in C.M. course

Contains $O(n,1)$ and also translations forming a group $\cong \mathbb{R}^{n,1}$. In fact, any $f \in IO(n,1)$ is of the form:

L is rotation, v translation

$$f(x) = Lx + v \quad \text{where } L \in O(n,1), \; v \in \mathbb{R}^{n+1}$$

We get
$$IO(n,1) \cong O(n,1) \times \mathbb{R}^{n,1} \quad \text{as } \underline{\text{sets}}:$$

$$f \longmapsto (L,v)$$

but it's NOT a direct product of groups.

Say f corresponds to (L,v), f' corresponds to (L',v').

$$f'(fx) = L'(Lx+v) + v' = \underbrace{L'L}x + \underbrace{L'v + v'}$$

If this were a direct product of groups, we'd get

So
$$(L',v')(L,v) = (L'L, L'v+v')$$

Thus, we have a <u>semi-direct</u> <u>product</u> using the fact that $O(n,1)$ acts as automorphisms of $\mathbb{R}^{n,1}$.

We say:
$$IO(n,1) = O(n,1) \ltimes \mathbb{R}^{n,1}$$

it points to what it acts on

In fact, group being acted on is a <u>normal</u> subgroup.
$H \triangleleft G$ (same triangle)

<u>Note</u>: $\mathbb{R}^{n,1} \triangleleft O(n,1)$ is a normal subgroup.

$O(n,1) \ltimes \mathbb{R}^{n,1}$ is too big since it's not connected (since $O(n,1)$ isn't connected).

Laws of physics aren't symmetrical under reflections!
($\exists$ particles that are diff. once we reflect them!)
$O(n,1)$ has 4 connected components — can switch past & future! reflection in time!

$O(n)$ has 2 connected components: $\det(f) = 1$
$\det(f) = -1$

reflection lives here

$O(n,1)$ has 4 connect. comps since we also have "time reversal":

$$(x_1, \ldots, x_{n+1}) \longmapsto (x_1, \ldots, x_n, -x_{n+1})$$

Pinors — rotations & reflections

Physics is not symmetrical under reflection or time reversal.

Let $O_0(n,1) \subseteq O(n,1)$ be the identity component and

$$IO_0(n,1) = O_0(n,1) \ltimes \mathbb{R}^{n,1}$$

is the identity component of Poincaré group.
(connected since product of connected spaces is connected)

But — $IO_0(n,1)$ is too small since spinors are not reps of $O_0(n,1)$ but only $Spin_0(n,1)$.

Recall: we have a 2-1 and onto homo

$$\tilde{\rho} : Spin_0(n,1) \longrightarrow O_0(n,1)$$

We can cook up a double cover of $IO_0(n,1)$ by replacing $O_0(n,1)$ with its double cover.

So — instead of $IO_0(n,1)$ we should really use

$$ISpin_0(n,1) = Spin_0(n,1) \ltimes \mathbb{R}^{n,1}$$

Note — $Spin_0(n,1)$ acts on $\mathbb{R}^{n,1}$ since $O_0(n,1)$ does. We just use $\tilde{\rho}$.

We're mostly interested in the case where $n=3$.

We want to know what the reps of $ISpin_0(n,1)$ are.

The full symmetry group could be bigger than $ISpin_0(n,1)$ but usually we just take a group like:

$$ISpin_0(n,1) \times G \quad \text{where } G \text{ is a compact Lie grp.}$$

We call this group the <u>internal</u> <u>symmetry group</u>.

We've got a classification of compact Lie groups.
(we know what their Lie algs are)

This is a direct product justified under some hypotheses by Coleman-Mandula Thm.

Different theories use different G's:

- For quantum electrodynamics (electromagnetism), $G = U(1)$ (related to electric charge)
- For quantum chromodynamics (color) (strong nuclear force), $G = SU(3)$ acts on $\mathbb{C}^3$
  call basis: red, green, & –
- For the electroweak force, the Glashow-Weinberg-Salam model

$$G = SU(2) \times U(1)$$

- For Standard Model:

$$G = \underbrace{SU(3)}_{\text{Chromodynamics}} \times \underbrace{SU(2) \times U(1)}_{\text{GWS model}}$$

So, in the Standard Model, all Hilbert spaces are reps of:

$$\underbrace{ISpin_0(3,1) \times SU(3) \times SU(2) \times U(1)}$$

symmetry group of all particles in nature.

In fact we use $G/\mathbb{Z}_6$; and reps of G are same as reps of this.

An elementary particle (in Standard model) will be an irreducible unitary rep. of this group, but not just any unitary irrep – only some appear in nature.

Recall – an irrep for a product is tensor product of irreps for each thing.

So, we need to know what irreps of

$$ISpin_0(3,1),\ SU(3),\ SU(2),\ U(1)\ \text{are.}$$

In fall we saw $\exists$ an irrep of $U(1)$ $\forall$ integer.

The relevant irreps of $SU(3)$, $SU(2)$, $U(1)$ are easy to understand, but what about $ISpin_0(3,1)$?

There are various sorts of unitary irreps, but for the Standard Model we only need 3 kinds:

<u>names for irreps:</u>

1) massive (or massless) spin-0 particle

2) massive (or massless) spin-½ particle
only ones that show up as elementary particles in Standard Model!

3) massless spin-1 particle

1) Let $\square = g^{ij} \partial_i \partial_j$ (∂_i means differentiate in i^{th} direction.)

and define the <u>mass-m, spin-0 particle</u> $(m>0)$ to be the space of solutions

$$\psi : \mathbb{R}^{n,1} \longrightarrow \mathbb{R}$$

of the Klein-Gordon equation: $(\square - m^2)\psi = 0$

How does the group $ISpin_0(n,1)$ act on this vector space of solns?

Problem: $(ff')\psi(x) = \psi(ff'x) = (f\psi)(f'x) = f'(f\psi)(x)$

argh! We need an inverse!

If ψ is a solution and $f = (L, v) \in IO_0(n,1)$
then

$f\psi$ is also a soln. where

guess: $f\psi(x) = \psi(fx)$ But this gives above prob.

So, we use $\boxed{f\psi(x) = \psi(f^{-1}x)}$

Since $\Box$ is defined using only g and since f preserves the metric, then

$$(\Box - m^2)(f\psi) = 0$$

Note — spin-½ particles are related to Clifford algs.

How can we solve the Klein-Gordon equation?

Try: A plain-wave soln.

$$\psi(\vec{x}, t) = e^{i(Et - \vec{p}\cdot\vec{x})}$$

Fix t, this is an exp. funct of x, so sines & cosines.

$x \in \mathbb{R}^{n,1}$, so $x = (\underset{\mathbb{R}^n}{\underset{\cap}{\vec{x}}}, \underset{\mathbb{R}}{\underset{\cap}{t}})$

Note: $\Box = g^{ij}\partial_i\partial_j = \nabla^2 - \dfrac{\partial^2}{\partial t^2}$

So to check $(\Box - m^2)\psi = 0$ note:

replace $\Box$ by $\nabla^2 - \dfrac{\partial^2}{\partial t^2}$

ψ - a real valued funct.

$$\nabla^2 \psi = -\overbrace{\vec{p}\cdot\vec{p}}^{p^2}\,\psi$$ derivs. in space direction

$$\frac{\partial^2}{\partial t^2}\psi = -E^2\psi$$ derivs. in time direction

So, $(\Box - m^2)\psi$ holds iff

$$(-p^2 + E^2 - m^2)\psi = 0$$

i.e. $-p^2 + E^2 - m^2 = 0$, or $\underline{E^2 = m^2 + p^2}$

This is famous in units where $c \neq 1$, where it becomes:

$$E^2 = m^2c^4 + p^2c^2$$ $\Big\{$ reln. bet energy of a particle & its momentum & mass.

This is the reln. between energy E, $\vec{p}$ momentum & mass m of a particle in special relativity.
If $\vec{p} = 0$,

$$E = mc^2.$$

take positive sqr. root

We should have a Hilbert space of these unitary reps, but we need to define the inner product!

We need to make the space of solns of the Klein-Gordon eqn. into a complex Hilbert space, i.e. define a complex structure (i.e. how to mult by i) and complex inner product and keep only those solns ψ w/ $\langle\psi,\psi\rangle < \infty$.

We also have to check that $IO_0(n,1)$ acts as <u>unitary</u> operators.

Using the homo $ISpin_0(n,1) \longrightarrow IO_0(n,1)$

we get a unitary rep of $ISpin_0(n,1)$. Now we have to check it's irreducible.

For spin-½ reps, we'll use the Dirac eqn. instead of the Klein-Gordon eqn.

$$(\not{\partial} + m)\psi = 0$$

has γ_i matrices

Now ψ is a pinor, or spin-valued funct. on $\mathbb{R}^{n,1}$.

For massless spin-1 rep, we'll use Maxwell's eqns:

We take 1-forms A on $\mathbb{R}^{n,1}$, let

$F = dA$, require Maxwell's eqns: $d*F = 0$.