"exists": preserves relns

"unique": generators

5/5/03

- trivial rep: each elt acts as ident.
- $$\begin{array}{ccc} U(1) & \hookrightarrow & GL(\mathbb{C}) \\ \alpha & \longmapsto & \alpha \end{array}$$ defining rep.

---

review of hmwk from last time: "Lie grp coincidences"

① Show C_n has a unique $*$-structure $*: C_n \circlearrowleft$ st $e_i^* = -e_i$.

pf: There is a unique antihomo $*: C_n \circlearrowleft$ w/ $e^* = -e_i$ because e_i generate C_n.

$$(\alpha a)^* = \alpha a^*$$
$$(a+b)^* = a^* + b^*$$
$$(ab)^* = b^* a^*$$

This antihomo exists because $*$ preserves the relations in C_n.

Reln: $e_i e_j + e_j e_i = 2\delta_{ij}$

so we need to check that

$$(e_i e_j + e_j e_i)^* = -2\delta_{ij} 1^*$$

follows from relns:

$$e_j^* e_i^* + e_i^* e_j^* = -2\delta_{ij} 1$$

$(-e_j)(-e_i) + (-e_i)(-e_j) = -2\delta_{ij} 1$ which is the Clifford alg. reln.

Finally, check $a^{**} = a \ \forall a \in C_n$. Follows from $e_i^{**} = e_i$, and the fact that e_i's generate C_n.

Note: $a^{**} = a$, $b^{**} = b$ $\Rightarrow$
$$(ab)^{**} = (b^*a^*)^* = (a^{**}b^{**}) = ab$$

and similarly for $(a+b)^{**}$, $(\alpha a)^{**}$; so once generators have $a^{**} = a$ we get it for all a.

Result of hmwk:

Thm – If G is a compact Lie group, then $\mathfrak{g}$ is isomorphic to a direct sum of copies of:

$$\mathfrak{u}(1),\ \underbrace{\mathfrak{so}(n),\ \mathfrak{su}(n),\ \mathfrak{sp}(n)}_{n \geq 2}$$

$\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ > related to octonians

This decomposition is unique modulo the following isos:

$$\mathfrak{so}(3) \cong \mathfrak{su}(2) \cong \mathfrak{sp}(1)$$

relnship bet. Pauli matrices & $\mathbb{H}$ (quaternions)

$\mathfrak{so}(4) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$

$\mathfrak{so}(5) \cong \mathfrak{sp}(2)$

$\mathfrak{so}(6) \cong \mathfrak{su}(4)$

Because of Bott periodicity, rotations in n-dim'l space repeats every 8 dimensions.

Spinors & Standard Model

Leptons & quarks — spin ½ reps.

Recall: Massive spin-0 rep: Take Minkowski spacetime

$$\mathbb{R}^{n,1} \cong \mathbb{R}^{n+1} \text{ w/ } g(v,w) = v_1 w_1 + \dots + v_n w_n - v_{n+1} w_{n+1}$$

and consider the space of "scalar fields"

$$\varphi : \mathbb{R}^{n,1} \longrightarrow \mathbb{R}$$

satisfying the Klein-Gordon eq: $(\Box - m^2)\varphi = 0$

where $\Box = g^{ab}\partial_a \partial_b$.

This is an irrep of $ISpin(n,1)$:

$$\begin{tikzcd}
\mathrm{Spin}_0(n,1) \ltimes \mathbb{R}^{n,1} \arrow[r, phantom, "\cong"] & ISpin_0(n,1) \arrow[r, hook, "\text{half-onto}"] \arrow[d, "2-1"] & ISpin(n,1) \arrow[r, hook, "\text{half-onto}"] \arrow[d, "2-1"] & IPin(n,1) \arrow[d, "2-1\ \tilde{\rho}"] \\
& ISO_0(n,1) \arrow[r, hook, "\text{half-onto}"] & ISO(n,1) \arrow[r, hook, "\text{half-onto}"] & IO(n,1)
\end{tikzcd}$$

$ISpin(n,1) \cong \mathrm{Spin}(n,1) \ltimes \mathbb{R}^{n,1}$, $\quad IPin(n,1) \cong \mathrm{Pin}(n,1) \ltimes \mathbb{R}^{n,1}$

$ISO_0(n,1) \cong SO_0(n,1) \ltimes \mathbb{R}^{n,1}$ → I (connected)

$ISO(n,1) \cong SO(n,1) \ltimes \mathbb{R}^{n,1}$ → I, PT (2 connect. comp)

P, T each have det = -1 since reflect 1 coord.

$IO(n,1) \cong O(n,1) \ltimes \mathbb{R}^{n,1}$ → I, Time reversal, Parity - reflects coor. of space, PT (4 connect. comp.)

* All grps on prev pg have same Lie alg.

<u>Note</u>: $ISpin_0(n,1)$ is connected & simply connected.

In fact, massive spin-0 particle is a rep of $IO(n,1)$ via:

$$(g\varphi)(x) = \varphi(g^{-1}x) \qquad x \in \mathbb{R}^{n,1}, \quad g \in O(n,1)$$

$IO(n,1)$ — all symmetries of spacetime.

<u>Massive spin-½ particle</u>

Form the Clifford algebra $C_{n,1}$ ($e_1, \ldots, e_{n+1}$) and pick a representation

$$\gamma : C_{n,1} \longrightarrow End(V)$$

for some v. space V (real or complex)

Given

$$\psi : \mathbb{R}^{n,1} \longrightarrow V \quad \text{we define}$$

$$\not{\partial}\psi : \mathbb{R}^{n,1} \longrightarrow V \quad \text{as follows:}$$

gamma matrices

Let $\gamma_i = \gamma(e_i) \in End(V)$ and

$$\gamma^i = g^{ij}\gamma_j \quad \text{(using Einstein sum convention)}$$

and note

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij}1_V$$

and thus $\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij} 1_V$

Then let $\boxed{\partial\!\!\!/ = \gamma^i \partial_i}$ (Feynman: $a\!\!\!/ = \gamma^i a_i$)

$\partial\!\!\!/$ is good because $\partial\!\!\!/^2 = (\gamma^i \partial_i)(\gamma^j \partial_j)$

$$= \gamma^i \gamma^j \underbrace{\partial_i \partial_j}_{\text{mixed partial}}$$

$$= \tfrac{1}{2}(\gamma^i\gamma^j + \gamma^j\gamma^i)\partial_i\partial_j$$

$$= \tfrac{1}{2}(2g^{ij} 1_V)\,\partial_i\partial_j$$

$$= g^{ij}\partial_i\partial_j = \Box.$$

The Dirac eqn. w/ mass $m \in \mathbb{R}$ is

$$(\partial\!\!\!/ + m)\psi = 0$$

This implies the Klein-Gordon eqn:

$$0 = (\partial\!\!\!/ - m)(\partial\!\!\!/ + m)\psi = (\partial\!\!\!/^2 - m^2)\psi = (\Box - m^2)\psi.$$

Particle physicists prefer the opposite sign of g and write the Klein-Gordon eqn as

$$(\Box + m^2)\psi = 0$$

But this forces our v. space to be complex!

A rep: $\rho: G \longrightarrow GL(V)$, so V is the space of solns and below is how a grp elt gives an elt of $GL(V)$

We have $(\Box + m^2) = (\not{\partial} + im)(\not{\partial} - im)$

but this only works if V is complex!

The space of solutions of $(\not{\partial} + m)\psi = 0$ is a rep. of $ISpin(n,1)$.

is a v. space so a rep.

How?

$$ISpin(n,1) = Spin(n,1) \ltimes \mathbb{R}^{n,1}$$

$$g \longmapsto (L, \underset{\text{translation}}{v})$$

Given a solution ψ of our eqn, so is $g\psi$ where

$$\underbrace{(g\psi)}_{=\,\rho(g)(\psi)}(\underset{\mathbb{R}^{n,1}}{x}) = L(\underbrace{\psi(\tilde{\rho}(g^{-1})x)}_{\in V})$$

$$ISpin(n,1) \xrightarrow[2\text{-}1]{\tilde{\rho}} ISO(n,1)$$

$ISO(n,1)$ acts as transf. of $\mathbb{R}^{n,1}$.

$L \in Spin(n,1)$, and acts on v. space V.

where L acts on V because

$C_{n,1}$ acts on V and $L \in Spin(n,1) \subseteq C^0_{n,1} \subset C_{n,1}$.

V is a rep of Clifford alg.

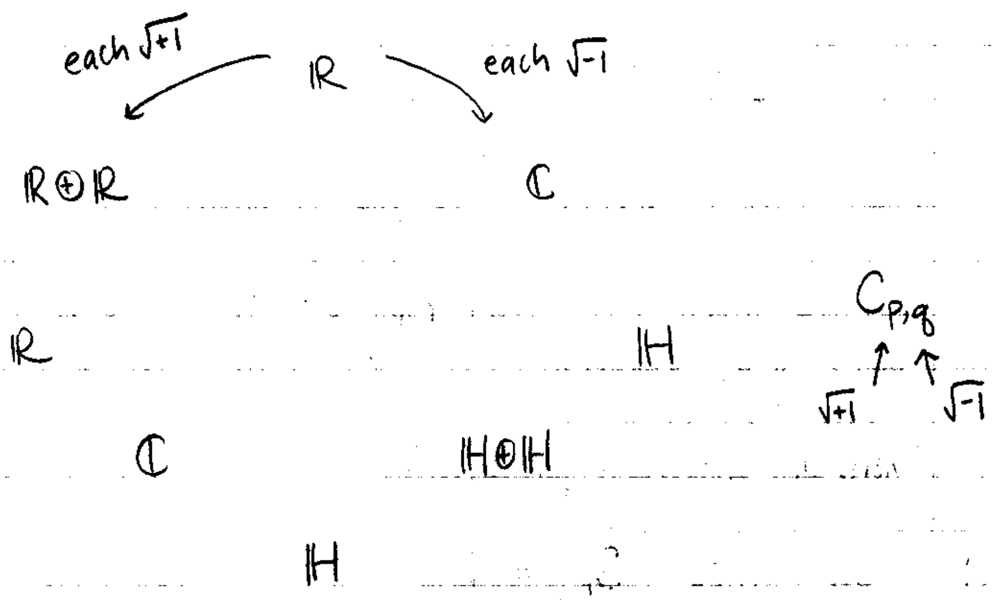
Therefore, $L \in C_{n,1}$, so knows how to act on guys in V.

$*$ <u>Check</u>: $(gg')\psi = g(g'(\psi))$.

If we take V to be an irrep of $C_{n,1}$, also known as a <u>pinor</u> rep, then we call the space of solns

$$\{ \psi : \mathbb{R}^{n,1} \longrightarrow V \mid (\not{\partial} + m)\psi = 0 \}$$

a "<u>mass-m spin-$\frac{1}{2}$ rep</u>" of $ISpin(n,1)$.



$C_{2,1} = \mathbb{R}\oplus\mathbb{R}$, $C_{6,1} = \mathbb{H}\oplus\mathbb{H}$

$C_{n,1}$ has <u>two</u> pinor reps if $n = 2, 6 \bmod 8$ and <u>one</u> otherwise.

We can also look at $\mathbb{C}_{n,1}$ and its <u>complex</u> irreps, aka complex pinors, and play the same game.

$$\begin{tikzcd} \mathbb{C} \arrow[d, bend right, "\text{each } \sqrt{+1}"'] \arrow[d, bend left, "\text{each } \sqrt{-1}"] \\ \mathbb{C}\oplus\mathbb{C} \end{tikzcd}$$

$C_{n,1}$ has two complex pinor reps if $n = 0 \bmod 2$, one otherwise.

Moral: For us, $n=3$, so we don't get two kinds of pinors, real or complex.

Massless spin-1/2 particle (e.g. all fermions in Standard model)

If $m=0$ the Dirac eqn.

$$\not{\partial}\psi = 0$$

makes sense even if ψ is just a spinor, not a pinor.

Spinors aren't necessarily reps of $C_{n,1}$, so what does $\gamma^j\gamma_j\psi$ mean?

There are different possibilities:

1) Sometimes $S_{n,1} \cong P_{n,1}$

i.e. we get spinors by restricting pinors to be rep of $\mathrm{Spin}(n,1) \subseteq \mathrm{Pin}(n,1)$

2) Sometimes $S_{n,1}^+ \oplus S_{n,1}^- \cong P_{n,1}$ and if

$a \in C_{n,1}^0$,

$$\gamma(a): S_{n,1}^{\pm} \longrightarrow S_{n,1}^{\pm} \quad \left(\text{sends } \begin{matrix} + \longmapsto + \\ - \longmapsto - \end{matrix}\right)$$

while if $a \in C_{n,1}^1$

$$\gamma(a): S_{n,1}^{\pm} \longrightarrow S_{n,1}^{\mp} \quad (+ \longmapsto -,\ - \longmapsto +)$$

We'll see this for $n=3$.

Recall—

$$\partial\!\!\!/ = \gamma^i \partial_i = g^{ij} \underbrace{\gamma(e_j)}_{\substack{\uparrow \\ C^1_{n,1}\text{-odd}}} \partial_i$$

so if $\psi : \mathbb{R}^{n,1} \longrightarrow S^{\pm}_{n,1}$ then

$$\partial\!\!\!/ \psi : \mathbb{R}^{n,1} \longrightarrow S^{\mp}_{n,1}$$

so $\partial\!\!\!/ \psi = 0$ makes sense.

We can't do this trick for $m \neq 0$, since

$$(\partial\!\!\!/ + m)\psi = 0$$

takes left to left handed spinors

takes left to right handed spinors

In this case we call

$$\{\psi : \mathbb{R}^{n,1} \longrightarrow S^{\pm}_{n,1} \mid \partial\!\!\!/ \psi = 0\}$$

the massless <u>left/right handed spin-½ rep. of</u> $I\mathrm{Spin}(n,1)$

All the fermions we see are of these 2 forms.
Again— we need to find a complex Hilbert space of solns to get a unitary rep of $I\mathrm{Spin}(n,1)$.

Let's try $n=3$.

$C_{3,1} \cong \mathbb{R}[4]$

dim $= 2^{3+1} = 16 = 4^2$, so 4×4 matrices

so pinors are:

called Majorana spinors

$P_{3,1} \cong \mathbb{R}^4$ (reps on $C_{3,1}$) (4x4 matrices act on col. vector of 4 entries)

and

$$\{\psi : \mathbb{R}^{3,1} \longrightarrow \mathbb{R}^4 \mid (\not{\partial} + m)\psi = 0\}$$

describes mass-m spin-$1/2$ particles which are their own antiparticles

- anti-particles – conjugate rep.
- rep = own conj. $\Leftrightarrow$ particle is its own antiparticle

Because this rep is $\cong$ to its conjugate rep since our fields take values in a real v. space.

Possibly neutrinos are Majorana spinors.

complexify: $\mathbb{C}_{3,1} \cong \mathbb{C} \otimes \mathbb{R}[4] \cong \mathbb{C}[4]$

so complex pinors are

$$P_{3,1}^{\mathbb{C}} \cong \mathbb{C}^4$$

Dirac spinors

$U(1)$ – describes electric charge!

and $\{\psi : \mathbb{R}^{3,1} \longrightarrow \mathbb{C}^4 \mid (\not{\partial} + m)\psi = 0\}$

describe mass-m spin-$1/2$ particles that aren't their own antiparticle; this is a rep of $U(1) \subseteq \mathbb{C}$ so these particles naturally have charge.

Even part:

$$C^0_{3,1} \cong C_{3,0} \quad \text{dim} = 8$$

$$\cong \mathbb{C}[2] \text{ — has } \dim_{\mathbb{C}} = 4, \text{ so } \dim_{\mathbb{R}} = 8.$$

so spinors are

$$S_{3,1} \cong \mathbb{C}^2, \quad \text{compare w/ } P_{3,1} \cong \mathbb{R}^4$$

thinking of as a complex v.space — real v.space

In fact, $S_{3,1}$ is just $P_{3,1}$ restricted to the spin group; now it can be made into a complex rep.

So – nothing really new.

$$\mathbb{C}^0_{3,1} \cong \mathbb{C} \otimes C_{3,1} \cong \mathbb{C} \otimes \mathbb{C}[2] \cong \mathbb{C}[2] \oplus \mathbb{C}[2]$$

so – we get 2 reps: left & right-handed complex spinors

are: $S_{3,1}^{\mathbb{C}\pm} \cong \mathbb{C}^2$ ← left and right handed Weyl spinors

$$\{\psi : \mathbb{R}^{3,1} \longrightarrow S_{3,1}^{\mathbb{C}\pm} \mid \not{\partial}\psi = 0\}$$

describes left/right handed massless spin-½ particles.

The latest version of the Standard Model has all fermions (quarks, leptons) as these.

$$\begin{array}{ccccc} \mathbb{C}^4 = P_{3,1}^{\mathbb{C}} \cong & S_{3,1}^{\mathbb{C}+} & \oplus & S_{3,1}^{\mathbb{C}-} \\ & \parallel & & \parallel \\ & \mathbb{C}^2 & & \mathbb{C}^2 \end{array}$$

What about $C_{1,3}$?

Only real pinors could possibly be different – everything else stays the same.

$C_{1,3} \cong \mathbb{H}[2]$

so, $P_{1,3} \cong \mathbb{H}^2$ (Different than $P_{3,1} \cong \mathbb{R}^4$)

But $\mathbb{H}^2$ resembles $\mathbb{C}^4$...

These give the same rep of $\mathrm{I}\mathrm{Spin}(3,1)$ as $P_{3,1}^{\mathbb{C}} \cong \mathbb{C}^4$

but for some reason now we're thinking of ψ as 2 quaternions instead of 4 complex numbers.

5/6/03 (notes on 4/29/03) QM

In CM — if your system is in some state, if you check to see if it's in another state — answer is always no. This is NOT true in QM — it can be in 2 states at once.

In QM — put a system in a state ψ, check to see if it's in the state ψ, answer is always yes. But this doesn't mean that if we check to see if it's in state φ answer is no!

$|\langle \varphi_i, \varphi_j \rangle|^2 = \delta_{ij}$, φ_i o.n. basis.

Observable — real-valued quantity we can measure about the system

(these correspond to self-adjoint operators on Hilb. space H.
$A: H \to H$

· We get different probabilities for each # when we measure an observable in some state.

Suppose φ_i is an o.n. basis st $A\varphi_i = a_i \varphi_i$, $a_i \in \mathbb{R}$

spectrum of possible values of the observable A in the state φ_i

If you measure A in some state ψ:

$$\psi = \sum \langle \varphi_i, \psi \rangle \varphi_i$$

Then the prob. of getting the result a_i is $|\langle \varphi_i, \psi \rangle|^2$.

$\langle \varphi_j, \varphi_i \rangle = \delta_{ij}$

cong. lin. lin.

$$\langle \psi, A\psi \rangle = \sum_{i,j} \langle \varphi_i, \psi \rangle \overline{\langle \varphi_j, \psi \rangle} \langle \varphi_j, a_i \varphi_i \rangle$$

$$= \sum_i \underbrace{|\langle \varphi_i, \psi \rangle|^2}_{\text{probability of measuring } a_i} a_i \qquad \langle \varphi_j, \varphi_i \rangle = \delta_{ij}$$

a_i: what we measure $\forall i$.

= average value of finding A in the state ψ
"expected value"

---

unitary operators on H — preserve all structure of Hilbert space.
We call these symmetries $U: H \longrightarrow H$. A symmetry maps states to states. (ex - move everything over 2 feet)
U - unitary means preserves inner products, so

$$\langle U\varphi, U\psi \rangle = \langle \varphi, \psi \rangle \quad \text{and}$$

$$|\langle U\varphi, U\psi \rangle|^2 = |\langle \varphi, \psi \rangle|^2$$

translation grp in $\mathbb{R}^3$, rotation grp in $\mathbb{R}^3$

A group of symmetries gives a unitary rep

$$\rho: G \longrightarrow U(H)$$

To classify the reps, we need only to classify the irreducible ones.

Physics | Math

Two systems

Two Hilbert spaces H, H'

"or" A system described using mutually exclusive possibilities (exclusive or)

$H \oplus H'$

Let $\{\varphi_i\}$ be an o.n. basis of H
$\{\varphi_j'\}$ " " H'

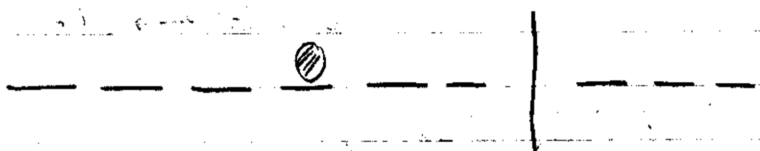
Then a basis for $H \oplus H'$ is $\{\varphi_i, \varphi_j'\}$

Combined system "and"
Some states of this system is a state of system 1 AND a state of system 2.

$H \otimes H'$
$\{\varphi_i \otimes \varphi_j'\}$

means if H has 3 ways of being, H' has 6, $H \otimes H'$ has 18 ways of being

$H \oplus H'$: H - ball in 1 of 6 boxes
H' - ball in 1 of 3 other boxes
$H \oplus H'$ - ball in 1 of 9 boxes



muffin/coffee — $H \oplus H' = 9$ ways
the states of system is 1 of 6 ways for muffin to be or 1 of 3 ways for coffee to be — so coffee/muffin can change into each other.

Ex) H = box w/ an electron
H' = box w/ a proton

$H \otimes H'$ = box w/ a proton & electron
$H \oplus H'$ = box w/ <u>either</u> a proton <u>or</u> electron in it

irreducible means — we can't describe the state as above as made up of either this or that

elementary goes along w/ direct sum

An <u>elementary system</u> w/ symmetry group G is an irreducible unitary rep of G. can't be describe as either "this or that"

So — an electron is an elementary particle means we can't describe it as either this or that!

Now — what is our group G?

$$G = G_1 \times \ldots \times G_n$$

an irrep of G is a tensor product of irreps of G_i.

$\rho : G \longrightarrow U(H)$ will be $H = H_1 \otimes \ldots \otimes H_n$ ← irreps
$\rho_i : G_i \longrightarrow U(H_i)$

$$\rho(g_1, \ldots, g_n) = \rho_1(g_1) \otimes \ldots \otimes \rho_n(g_n).$$

What's an electron? An irrep of G, meaning made up of irreps of each G_i.

SU(2), SL(2, $\mathbb{C}$) has 1 n-dim'l irrep $\forall n$, which we call spin-j irrep where $j = n/2$.

electron – have right & left handed – this tells us the way it spins – they're both just electrons, but simply spin opposite ways.

ex) right-handed electron:

$$\psi : \mathbb{R}^{3,1} \longrightarrow S_{3,1}^{\mathbb{C}^-}$$

$$H = \{\psi : \mathbb{R}^{3,1} \longrightarrow \mathbb{C}^2 \text{ st } \cancel{\partial}\psi = 0\} \otimes \underbrace{\mathbb{C}_1 \otimes \mathbb{C} \otimes \mathbb{C}_{-2}}_{\text{are 1-dim'l}}$$

Hilbert space of right-handed electron

so H is just space of solns.

$$J_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ ang. mom. in } z\text{-direction} \quad J_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{Pauli matrices}$$

(Miguel) 5/6/03 The Weyl algebra & Fock space

Quantization of 1D systems (eg particle on a line)

configuration space

The classical phase space of a particle on a line is $P = \mathbb{R}^2$ w/ coordinates p & q
momentum — position

phase space — position & momentum

Phase space has a Poison bracket $\{p,q\} = 1$ [1]
(see CM notes Fall 2002) [0] derivation, antisymmetric, [2] $\mathbb{R}$-linear on space of functs on phase space
and satisfies Jacobi iden. [3]
(derivation) $\{f, gh\} = \{f,g\}h + g\{f,h\}$

We get $C^\infty(P)$ is a Lie alg. (by 1-3) and in fact a Poisson alg. (adding in 0)

A classical mechanical system is a manifold P & a Poisson structure on $C^\infty(P)$.
According to Heisenberg to quantize this system we replace the classical "algebra of observables" $C^\infty(P)$ by an algebra of operators on a Hilbert space.

Dirac quantization prescription:

functions $f \longmapsto$ operators $\hat{f}$ we require

$$\hat{f}\hat{g} - \hat{g}\hat{f} =: [\hat{f}, \hat{g}] = -i\hbar \underbrace{\widehat{\{f,g\}}}_{\text{another operator}}$$

Planck's constant

We'd like to do this, but it's impossible!

$fg = gf$ for functions, but not operators $\hat{f}\hat{g} \neq \hat{g}\hat{f}$.

We'll content ourselves w/ doing $f \longrightarrow \hat{f}$ when f is a linear function on P. P was a manifold, now is a vector space.

We'll concentrate on $\left.\begin{array}{l} q \longmapsto \hat{q} \\ p \longmapsto \hat{p} \end{array}\right\}$ operators on some Hilbert space.

Since $\{p, q\} = 1$ then by Dirac $[\hat{p}, \hat{q}] = -i\hbar \hat{1}$.

Schrödinger representation: $\left(\mathcal{O} \longrightarrow \mathrm{End}(H)\right)$ alg. of observables

Is $H = L^2(\mathbb{R})$ this is the line that the particle lives in.

$\hat{q} : \psi \longmapsto (\hat{q}\psi)(x) = x\psi(x)$

$(\hat{p}\psi)(x) = -i\hbar \dfrac{\partial \psi}{\partial x}$ $\qquad$ $\hat{p}$ & $\hat{q}$ are densely defined unbounded operators on H.

ex) $C_0^\infty(\mathbb{R}) = \{$compactly supported C^∞-functs$\}$

$\overline{C_0^\infty(\mathbb{R})} = L^2(\mathbb{R})$

$[\hat{p}, \hat{q}] = -i\hbar \hat{1}$ wants to eat a function...

$$[\hat{p}, \hat{q}]\psi(x) = \hat{p}\hat{q}\psi(x) - \hat{q}\hat{p}\psi(x) = \hat{p}\, x\psi(x) + \hat{q}\, i\hbar \frac{\partial \psi(x)}{\partial x}$$

$$= -i\hbar \frac{\partial}{\partial x}(x\psi(x)) + x i\hbar \frac{\partial \psi(x)}{\partial x}$$

$$= -i\hbar \psi(x)$$

$$= -i\hbar (\hat{1}\psi)(x)$$

Operators: · densely defined & bounded $\Rightarrow$ extends by continuity to everywhere defined

· everywhere defined & unbounded requires nonconstructive method

$(P, C^\infty(P), \{\cdot,\cdot\}) \rightsquigarrow$ (P is a v. space, P^* is a space of linear functions, W is a symplectic bilinear form on P^*)

$$W: P^* \times P^* \longrightarrow \mathbb{R}$$

linear functs $(f, g) \longmapsto \{f, g\}$

Poisson bracket of linear functs is a multiple of 1. In general, $\{\cdot,\cdot\}$ of 2 functs is a funct. But if they're linear & we assume $\{p, q\} = 1$ we get the $\{\cdot,\cdot\} \in \mathbb{R}$.

A symplectic form on a v. space V is a skew-symmetric, non-degenerate bilinear form on V.

similar to a metric on V.

$\omega(x, y) + \omega(y, x) = 0 \quad \forall\, x, y \in V$ (skew-symm)

if $\omega(x, y) = 0 \quad \forall\, y \in V$ then $x = 0$. (non-deg)

$\omega(ax+by, z) = a\omega(x, z) + b\omega(y, z)$ (bilinear form)

$$\{p, q\} = 1 \text{ implies } \{p, f(q)\} = f'(q) \text{ if } f \text{ is analytic}$$

A vector space w/ a symplectic form on it is called a symplectic vector space.

If (V,ω) is a symplectic v. space, we define the Weyl alg on (V,ω):

$$W(V,\omega) := T(V) \Big/ {x\otimes y - y\otimes x = i\hbar\,\omega(x,y)\,1}$$

This is the alg. generated by V modulo the Heisenberg relns.

$$[\hat{x},\hat{q}] = -i\hbar\,\omega(x,y)\,\hat{1}$$

Compare to Clifford alg on (V,g): If $\dim(V) = n$, $\dim C(V,g) = 2^n$.

$$C(V,g) := T(V) \Big/ {x\otimes y + y\otimes x = 2g(x,y)}$$

But $W(V,\omega)$ is infinite dim'l even if $\dim V = 2$.

Let $\dim V = 2$, p, q form a basis for V so we get:

$$1,\ p,\ q,\ p^2,\ pq,\ qp,\ q^2, \ldots$$

$pq = qp - i\hbar\,\omega(p,q)$, so pq, qp are linearly dependant

so we get: $1, p, q, p^2, qp, q^2, p^3, qp^2, q^2p, q^3, \ldots$

so, $W(V,\omega)$ is iso as v. spaces the space of polys on (p,q) w/ a "twisted" product $qp \neq pq$.
(we put all q's in front of all p's).

Thm - If $\hat{x}, \hat{y}$ are elements of a Banach algebra then

$$[\hat{x}, \hat{y}] \neq \hat{1}$$

(we can't put a finite norm on a repreentation of the Heisenberg relns)

A Banach algebra is an algebra w/ a norm $\|\cdot\|$ defined on it st $\|ab\| \le \|a\| \|b\|$ and it's complete w/r/t norm topology.

pf: Assume $[\hat{x}, \hat{y}] = \hat{1}$ and $y \neq 0$, then $[x, y^n] = n y^{n-1}$ & $y^n = 0$ by induction (x acts like a deriv. of functs of y)

$$n y^{n-1} = x y^n - y^n x,$$

$$n \| y^{n-1} \| = \| x y^n - y^n x \| \le \| x y^n \| + \| y^n x \| \le 2 \|x\| \|y\| \|y^{n-1}\|$$

so we can divide through by $\|y^{n-1}\|$ and obtain:

$$n \le 2 \|x\| \|y\| \quad \forall n \quad \rightarrow\leftarrow . \ \blacksquare$$

We can fix this by considering $A \longmapsto e^{iA}$ which maps possibly unbounded self-adjoint operators to unitary operators (w/ norm = 1)

We had $V = \mathbb{R}^2 \ni (p, q)$

$$(\hat{q} \psi)(x) = x \psi(x), \quad (\hat{p} \psi)(x) = -i\hbar \frac{d\psi}{dx}(x) \text{ on } L^2(\mathbb{R})$$

$$\left.\begin{aligned} [e^{ik\hat{q}} \psi](x) &= e^{ikx} \psi(x) \\ [e^{i a/\hbar \, \hat{p}} \psi](x) &= \psi(x-a) \end{aligned}\right\} \text{unitary operators}$$

* the exponential of the derivatives is translation by Taylor's Thm

Let $\varphi : V \longrightarrow W(V,\omega)$
$x \longmapsto x$ (inclusion) (linear)

Let $\mathcal{W} : V \longrightarrow \mathcal{U}(H)$ – unitary operators on H
$x \longmapsto e^{i\varphi(x)}$
$kx \longmapsto e^{i\varphi(kx)} = e^{ik\varphi(x)}$

(*) What does $\varphi(x)\varphi(y) - \varphi(y)\varphi(x) = -i\hbar\omega(x,y)$ imply for $\mathcal{W}(x)$ and $\mathcal{W}(y)$?

Thm – (<u>Baker-Campbell-Hausdorff</u>): If $[x,y]$ commutes w/ x and y then $e^x e^y = e^{(x+y+\frac{1}{2}[x,y])}$

If we apply this to $e^{i\varphi(x)}$... we get

$$\mathcal{W}(x)\mathcal{W}(y) = e^{i\varphi(x)}e^{i\varphi(y)} = e^{i(\varphi(x)+\varphi(y)) + \frac{1}{2}[i\varphi(x), i\varphi(y)]}$$

φ is linear
$$= e^{i\varphi(x+y) - \frac{1}{2}[\varphi(x),\varphi(y)]}$$

by (*)
$$= \mathcal{W}(x+y)e^{i\hbar/2\,\omega(x,y)}$$

A <u>Weyl system</u> on (V,ω) is a map $\mathcal{W} : V \longrightarrow \mathcal{U}(H)$ such that

$$\mathcal{W}(x)\mathcal{W}(y) = \mathcal{W}(x+y)e^{i\hbar/2\,\omega(x,y)}$$

<u>Fock Space</u>

In 1-dim we had $\hat{p}, \hat{q}$, $[\hat{p},\hat{q}] = -i\hbar$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{1}{\lambda}\hat{q} + \frac{i\lambda}{\hbar}\hat{p}\right)$$

$\hat{a}$ has an adjoint

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(\frac{1}{\lambda}\hat{q} - \frac{i\lambda}{\hbar}\hat{p}\right)$$, chosen so that $[\hat{a}, \hat{a}^+] = 1$.

called the number operator

$\hat{N} = \hat{a}^+\hat{a}$ is positive, self-adjoint and its spectrum is $\mathbb{N}$

$\hat{N}$ has eigenvectors, one of them is zero: $|0\rangle$ and satisfies

$\hat{N}|0\rangle = 0$

$\hat{a}|0\rangle = 0 \longrightarrow \hat{a}$ has one eigenvector $\forall z \in \mathbb{C}$

$\hat{a}|z\rangle = z|z\rangle$

$\langle z|w\rangle \neq 0$ for any $z, w \in \mathbb{C}$

$$k\int_{\mathbb{C}} |z\rangle\langle z|\, dz\, d\bar{z} \propto 1$$

For $v \in H$, e is a vector of norm 1, then $|e\rangle\langle e, v\rangle$ is the projection of v on the line spanned by e.

If $|\psi\rangle$ is any state in H then $\psi(z) = \langle z, \psi\rangle$ is a "wave function". Every state in H gets represented as a function from $\mathbb{C}$ to $\mathbb{C}$.

$$\langle \psi, \varphi\rangle = k\int_{\mathbb{C}} \langle \psi, z\rangle\langle z, \varphi\rangle\, dz\, d\bar{z}$$

$$= k\int_{\mathbb{C}} \psi^*(z)\,\varphi(z)\, dz\, d\bar{z}$$