5/5/03

- trivial rep: each elt acts as ident.
  \[ \begin{align*}
  U(1) & \longrightarrow GL(\mathbb{C}) \\
  \alpha & \longrightarrow \alpha 
  \end{align*} \]
  defining rep.

- review of hmwk from last time: Lie grp coincidences

Show \( C_n \) has a unique \(-\) structure \( \ast : C_n \, 5 \) st
\( e_i \ast = -e_i \).

**pf:** There is a unique antihomo \( \ast : C_n \, 5 \) wth \( e^\ast = -e_i \)
because \( e_i \) generate \( C_n \).

\[
\begin{align*}
(a^\ast)^\ast &= a^\ast \\
(a + b)^\ast &= a^\ast + b^\ast \\
(ab)^\ast &= b^\ast a^\ast
\end{align*}
\]

This antihomo exists because \( \ast \) preserves the relations in \( C_n \).

Reln: \( e_i e_j + e_j e_i = 2 \delta_{ij} \).

so we need to check that
\[ (e_i e_j + e_j e_i)^\ast = -2 \delta_{ij} \ast 1^\ast \]
follows from relns:
\[ e_j^\ast e_i^\ast + e_i^\ast e_j^\ast = -2 \delta_{ij} \ast 1 \]
\[ (-e_j)(-e_i) + (-e_i)(-e_j) = -2 \delta_{ij} \ast 1 \text{ which is the Clifford alg. reln.} \]
Finally, check \( a^{**} = a \) for all \( a \in C_n \). Follows from \( e_{i}^{**} = e_{i} \), and the fact that \( e_{i} \)'s generate \( C_n \).

Note: \[
\begin{align*}
  a^{**} &= a \\
  b^{**} &= b \\
  (ab)^{**} &= (b^*a^*)^* \\
  &= (a^{**} b^{**}) \\
  &= ab
\end{align*}
\]

and similarly for \( (a + b)^{**} \), \( \alpha a^{**} \); so once generators have \( a^{**} = a \) we get it for all \( a \).

Result of homework:

Thm: If \( G \) is a compact Lie group, then \( g \) is isomorphic to a direct sum of copies of:

\[
\begin{align*}
  u(1), \; so(n), \; su(n), \; sp(n) \\
  \text{for } n \geq 2
\end{align*}
\]

\( g_2, f_4, e_6, e_7, e_8 \) related to octonions

This decomposition is unique modulo the following isos:

\[
\begin{align*}
  so(3) &\cong su(2) \cong sp(1) \\
  so(4) &\cong sp(1) \oplus sp(1) \\
  so(5) &\cong sp(2) \\
  so(6) &\cong su(4)
\end{align*}
\]

Relation between Pauli matrices \( q \) and \( IH \) (quaternions)
Because of Bott periodicity, rotations in $n$-dim'l space repeats every 8 dimensions.

**Spinors & Standard Model**

Leptons $\Psi$, quarks $\Psi_{1/2}$ reps.

Recall: Massive spin-$0$ rep: Take Minkowski spacetime

$$\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1} \text{ w/ } g(v,w) = v_1w_1 + \ldots + v_nw_n - v_{n+1}w_{n+1}$$

and consider the space of "scalar fields"

$$\Psi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$$

satisfying the Klein–Gordon eq:

$$(\Box - m^2) \Psi = 0$$

where $\Box = g^{ab} \partial_a \partial_b$.

This is an irrep of $\text{I Spin}(n,1)$:

$$\text{I Spin}_0(n,1) \overset{\text{half-onto}}{\rightarrow} \text{I Spin}(n,1) \overset{\text{half-onto}}{\rightarrow} \text{I Spin}(n,1)$$

$$\text{Spin}(n,1) \times \mathbb{R}^{n+1}$$

$$\text{SO}_0(n,1) \overset{\text{half-onto}}{\rightarrow} \text{SO}(n,1) \overset{\text{half-onto}}{\rightarrow} \text{O}(n,1)$$

$$\text{SO}(n,1) \times \mathbb{R}^{n+1}$$

$I$, $\text{I, PT}$ have det $=-1$ since reflect 1 coord.

$I$, Time reversal, parity - reflects coord. of space, $\text{PT}$ (4 connect. comp.)
* All grps on prev pg have same Lie alg.

**Note:** $I Spin_o(n,1)$ is connected or simply connected.

In fact, massive spin-0 particle is a rep of $I O(n,1)$ via:

$$(g \phi)(x) = \phi(g^{-1}x) \quad x \in \mathbb{R}^{n,1}$$

$g \in O(n,1)$

$I O(n,1)$ — all symmetries of spacetime.

---

**Massive spin-$\frac{1}{2}$ particle**

Form the Clifford algebra $C_{n,1}^{e_1,\ldots,e_n}$ and pick a representation:

$$\gamma : C_{n,1} \rightarrow \text{End} \left( V \right)$$

for some vector space $V$ (real or complex)

Given

$$\gamma : \mathbb{R}^{n,1} \rightarrow V$$

we define

$$\gamma \gamma : \mathbb{R}^{n,1} \rightarrow V$$

as follows:

Let $\gamma_i = \gamma(e_i) \in \text{End} \left( V \right)$ and

$$\gamma^i = g^{ij} \gamma_j$$

(using Einstein sum convention)

and note

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2 g_{ij} 1_V$$
and thus \( \delta_i \delta_j + \delta_j \delta_i = 2g \delta_i \delta_j \).

Then let \( \gamma = \delta_i \delta_i \) (Feynman: \( \gamma = \delta_i \delta_i \)).

\( \gamma \) is good because \( \gamma^2 = (\delta_i \delta_j)(\delta^i \delta^j) \)

\[
= \delta_i \delta_j \delta^i \delta^j \\
= \frac{1}{2} \left( \delta_i^2 \delta^i + \delta^i \delta^i \right) \delta_i \delta_j \\
= \frac{1}{2} \left( 2g \delta_i \delta_j \right) \delta_i \delta_j \\
= g \delta_i \delta_i \delta_j = \Box.
\]

The Dirac eqn. w/ mass \( m \in \mathbb{R} \) is

\((\gamma + m) \psi = 0 \)

This implies the Klein–Gordon eqn:

\( 0 = (\gamma - m)(\gamma + m) \psi = (\gamma^2 - m^2) \psi = (\Box - m^2) \psi. \)

Particle physicists prefer the opposite sign of \( g \) and write the Klein–Gordon eqn as

\((\Box + m^2) \psi = 0 \)

But this forces our \( \nu \) space to be complex!
A rep: \( \rho: G \rightarrow \text{GL}(V) \), so \( V \) is the space of solns.

and below is how a grp elt gives an elt of \( \text{GL}(V) \).

We have \( (D+m^2) = (\mathcal{F} + im)(\mathcal{F} - im) \)

but this only works if \( V \) is complex!

The space of solutions of \((\mathcal{F} + m)Y = 0\) is a rep. of \( \text{Ispin}(n,1) \).

How?

\[
\text{Ispin}(n,1) = \text{Spin}(n,1) \rtimes \mathbb{R}^{n+1}
\]

\[
g \xrightarrow{} (L, \nu) \quad \text{translation}
\]

Given a solution \( Y \) of our eqn, so is \( gY \)

where \( (gY)(x) = L\left( Y\left( \rho(g^{-1})x \right) \right) \).

\[
\rho(gY) \quad \mathbb{R}^{n+1} \xrightarrow{\nu} V \\
\text{Ispin}(n,1) \xrightarrow{\sim} \text{ISO}(n,1)
\]

where \( L \) acts on \( V \) because \( C_{n,1} \) acts on \( V \) and \( L \in \text{Spin}(n,1) \subseteq C_{n,1}^0 \subseteq C_{n,1} \).

Therefore, \( L \in C_{n,1} \), so knows how to act on guys in \( V \).

\[
* \quad \text{Check: } (gg')Y = g(g'(Y)).
\]
If we take $V$ to be an irrep of $C_{n,1}$, also known as a spinor rep, then we call the space of solns

$$\{ \psi: \mathbb{R}^{n_1} \rightarrow V \mid (\xi + m) \psi = 0 \}$$

a "mass-$m$ spin-$\frac{1}{2}$ rep" of $I\text{Spin}(n,1)$.

$$\begin{array}{ccc}
\mathbb{R} & \mathbb{R} & \mathbb{C} \\
\mathbb{R} & \mathbb{H} & \mathbb{C}_p,q \\
\mathbb{C} & \mathbb{H} \oplus \mathbb{H} & \\
\mathbb{H} & & \\
C_{2,1} = \mathbb{R} \oplus \mathbb{R}, & C_{6,1} = \mathbb{H} \oplus \mathbb{H} & \\
C_{n,1} \text{ has two spinor reps if } n = 2, 6 \text{ mod 8 and one otherwise.} & & \\
& \text{We can also look at } C_{n,1} \text{ and its complex irreps, aka complex spinors, and play the same game.} \\
& C_{n,1} \text{ has two complex spinor reps if } n = 0 \text{ mod 2, one otherwise.}
\end{array}$$
Moral: For us, $n=3$, so we don’t get two kinds of spinors, real or complex.

**Massless spin-$\frac{1}{2}$ particle** (e.g. all fermions in Standard model)

If $m=0$ the Dirac eqn.

$$\gamma^0 = 0$$

makes sense even if $\gamma^0$ is just a spinor, not a pinor.

Spinors aren’t necessarily reps of $C_{n,1}$, so what does $\gamma^0, \gamma_i$ mean?

There are different possibilities:

1) Sometimes $S_{n,1} \cong P_{n,1}$

i.e. we get spinors by restricting pinors to be rep of Spin($n,1) \subseteq Pin(n,1)$

2) Sometimes $S_{n,1}^+ \oplus S_{n,1}^- \cong P_{n,1}$ and if $a \in C_{n,1}^0$,

$$\delta(a): S_{n,1}^\pm \rightarrow S_{n,1}^\pm \quad (\text{ sends } + \rightarrow +)$$

while if $a \in C_{n,1}^1$,

$$\delta(a): S_{n,1}^\pm \rightarrow S_{n,1}^\pm \quad (+ \rightarrow -, - \rightarrow +)$$

We’ll see this for $n=3$. 
Recall:

\[ \gamma = \gamma^i \delta_i = g^{ij} \delta(e_j) \delta_i \]

so if \( Y: \mathbb{R}^n \rightarrow S^n_{(n)} \), then

\[ \gamma Y: \mathbb{R}^n \rightarrow S^n_{(n)} \]

so \( \gamma Y = 0 \) makes sense.

We can't do this trick for \( m \neq 0 \), since

\[ (\gamma + m) Y = 0 \]

\[ \text{takes left to right handed spinors} \]

In this case we call

\[ \{ Y: \mathbb{R}^n \rightarrow S^n_{(n)} \mid \gamma Y = 0 \} \]

the massless left/right handed spin-\( \frac{1}{2} \) rep. of \( \mathbb{I} \text{Spin}(n,1) \).

All the fermions we see are of these 2 forms.

Again— we need to find a complex Hilbert space of solutions to get a unitary rep of \( \mathbb{I} \text{Spin}(n,1) \).

Let's try \( n = 3 \).
\[ C_{3,1} \cong \mathbb{R}[4] \]

\[ \dim = 2^{3+1} = 16 = 4^2, \text{ so } 4 \times 4 \text{ matrices} \]

So spinors are:

\[ P_{3,1} \cong \mathbb{R}^4 \] (reps on \( C_{3,1} \)) \((4 \times 4 \text{ matrices act on col. vector of 4 entries})\)

and \[
\begin{Bmatrix}
4: \mathbb{R}^{3,1} \rightarrow \mathbb{R}^4 \mid (\gamma + m) \psi = 0
\end{Bmatrix}
\]
describes mass-m spin-\(\frac{1}{2}\) particles which are their own antiparticles.

- anti-particles - conjugate rep.
- rep = own conj. \(\iff\) particle is its own antiparticle

Because this rep is \(\cong\) to its conjugate rep since our fields take values in a real v. space.

Possibly neutrinos are Majorana spinors.

Complexify:

\[ C_{3,1} \cong \mathbb{C} \otimes \mathbb{R}[4] \cong \mathbb{C}[4] \]

so complex spinors are:

\[ P_{3,1}^c \cong \mathbb{C}^4 \]

Dirac spinors
\( U(1) \) describes electric charge.

and \( \{ 4 : \mathbb{R}^{3,1} \to \mathbb{C}^4 \mid (\gamma + m) 4 = 0 \} \)

describe mass-\( m \) spin-\( \frac{1}{2} \) particles that aren't their own antiparticles; this is a rep of \( U(1) \). So these particles naturally have charge.

Even part:

\[
C_{3,1}^0 \cong C_{3,0} \quad \text{dim} = 8
\]

\[
\cong C[2] \quad \text{has dim}_c = 4 \text{, so dim}_r = 8.
\]

So spinors are

\[
S_{3,1} \cong \mathbb{C}^2, \quad \text{compare w/} \quad P_{3,1} \cong \mathbb{R}^4.
\]

thinking of as

\[ \uparrow \text{complex vs. space} \]

\[ \uparrow \text{real vs. space} \]

In fact, \( S_{3,1} \) is just \( P_{3,1} \) restricted to the spin group; now it can be made into a complex rep.

So - nothing really new.

\[
C_{3,1}^0 \cong C \otimes C_{3,1} \cong C \otimes C[2] \cong C[2] \otimes C[2]
\]

So - we get 2 reps: left \( \uparrow \) right-handed complex spinors

are:

\[
S_{3,1}^\pm \cong \mathbb{C}^2 \quad \text{left and right-handed Weyl spinors}
\]
\{ \psi : \mathbb{R}^{3,1} \rightarrow S_{3,1}^c \ | \ \psi \psi^* = 0 \}

describes left/right handed massless spin-\frac{1}{2} particles.

The latest version of the Standard Model has all fermions (quarks, leptons) as these.

\[ C^4 = P_{3,1}^c \cong S_{3,1}^c \oplus S_{3,1}^c \]

What about \( C_{1,3} \)?

Only real spinors could possibly be different — everything else stays the same.

\[ C_{1,3} \cong \mathbb{H}[2] \]

so, \( P_{1,3} \cong \mathbb{H}^2 \) (Different than \( P_{3,1} \cong \mathbb{R}^4 \))

But \( \mathbb{H}^2 \) resembles \( C^4 \).

These give the same rep of \( \text{I} \text{Spin}(3,1) \) as \( P_{3,1}^c \cong C^4 \)

but for some reason now we're thinking of \( \psi \) as 2 quaternions instead of 4 complex numbers.
In QM— if your system is in some state, if you check to see if it's in another state — answer is always no. This is NOT true in QM — it can be in 2 states at once.

In QM— put a system in a state \( \psi \), check to see if it's in the state \( \psi \) answer is always yes. But this doesn't mean that if we check to see if it's in state \( \psi \) answer is no!

\[
|\langle \psi_i, \psi_j \rangle|^2 = \delta_{ij}, \quad \psi_i \text{ o.n. basis.}
\]

**Observable** — real-valued quantity we can measure about the system (these correspond to self-adjoint operators on Hilbert space \( \mathcal{H} \)).

We get different probabilities for each \( \psi \) when we measure an observable in some state.

Suppose \( \psi_i \) is an o.n. basis s.t. \( A\psi_i = a_i \psi_i, \ a_i \in \mathbb{R} \)

\( \uparrow \) spectrum of possible values of the observable \( A \) in the state \( \psi_i \).

If you measure \( A \) in some state \( \psi \):

\[
\psi = \sum \langle \psi_i, \psi \rangle \psi_i
\]

Then the prob. of getting the result \( a_i \) is \( |\langle \psi_i, \psi \rangle|^2 \).
\[ \langle \psi', \psi \rangle = \delta_{ij} \]

\[ \langle 4, A4 \rangle = \sum_{ij} \langle \psi_i, 4 \rangle \langle \phi_j, 4 \rangle \langle \phi_j, \psi_i \rangle \]

\[ = \sum_i |\langle \psi_i, 4 \rangle|^2 q_i \]

\[ \text{probability of measuring } q_i \text{ what we measure } A_i \]

\[ = \text{average value of finding } A \text{ in the state } 4 \]

"expected value"

Unitary operators on \( H \) preserve all structure of Hilbert space.
We call these symmetries \( U : H \rightarrow H \). A symmetry maps states to states. (ex: move everything over 2 feet)
\( U \) - unitary means preserves inner products, so

\[ \langle U\psi, U\psi \rangle = \langle \psi, \psi \rangle \quad \text{and} \]

\[ \langle U\psi, U\psi \rangle^2 = |\langle \psi, \psi \rangle|^2 \]

A group of symmetries gives a unitary rep \( \rho : G \rightarrow U(H) \)

To classify the reps, we need only to classify the irreducible ones.
Physics

Two systems

"or" : A system described using mutually exclusive possibilities (exclusive or)

Math

Two, Hilbert spaces $H, H'$

Let $\{q_i\}$ be an o.n. basis of $H$

Then a basis for $H \otimes H'$ is $\{q_i, q_j'\}$

Combined system "and"

Some states of this system is a state of system 1 and a state of system 2.

H$\otimes$H' : H - ball in 1 of 6 boxes

H' - ball in 1 of 3 other boxes

H $\otimes$ H' - ball in 1 of 9 boxes

muffin/coffee $-$ H$\otimes$H' = 9 ways

the states of system is 1 of 6 ways for muffin to be or 1 of 3 ways for coffee to be - so coffee/muffin can change into each other.
Ex) $H = \text{box w/ an electron}$
$H' = \text{box w/ a proton}$

$H \otimes H' = \text{box w/ a proton + electron}$
$H \otimes H' = \text{box w/ either a proton or electron in it}$

Irreducible means—we can't describe the state as above as either this or that made up of.

An elementary system w/ symmetry group $G_i$ is an irreducible unitary rep of $G_i$. Can't be describe as either "this or that".

So— an electron is an elementary particle means we can't describe it as either this or that!

Now— what is our group $G_i$?

$G_i = G_{i_1} \times \ldots \times G_{i_n}$.

An irrep of $G_i$ is a tensor product of irreps of $G_{i}$. \( \rho: G_i \rightarrow U(\mathbb{C}) \) will be $H = H_1 \otimes \ldots \otimes H_n \in \text{irreps}$

$\rho(g_1, \ldots, g_n) = \rho_1(g_1) \otimes \ldots \otimes \rho_n(g_n)$.

What's an electron? An irrep of $G_i$, meaning made up of irreps of each $G_{i_j}$. 
SU(2), SL(2, C) has n-dim irrep \( V_n \), which we call spin-\( j \) irrep where \( j = n/2 \).

Electrons have right-\( j \), left-handed — this tells us the way it spins — they're both just electrons, but simply spin opposite ways.

ex) right-handed electron:

\[ Y : \mathbb{R}^3 \rightarrow S^2 \]

\[ H = \{ Y : \mathbb{R}^3 
\rightarrow \mathbb{C}^2 \quad \text{st.} \quad Y = 0 \}\]

Hilbert space of right-handed electron is 1-dim.

Hence, it's just space of solutions.

\[ J_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{any mom in z-direction} \]

\[ J_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{Poincari matrices} \]

\[ J_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]
5/4/03 The Weyl algebra & Fock space

Quantization of 10 systems (e.g., particle on a line)

The classical phase space of a particle on a line is \( \mathbb{R}^2 \) w/ coordinates \( p, q \):
- momentum
- position

Phase space — position \( q \), momentum \( p \)

Phase space has a Poisson bracket \( \{ p, q \} = 1 \) (see CM notes Fall 2002)
- derivation, antisymmetric, \( \mathbb{R} \)-linear

on space of functions on phase space

and satisfies Jacobi iden. \( \{ f, gh \} = \{ f, g \} h + g \{ f, h \} \)

We get \( C^\infty(P) \) is a Lie alg., and in fact a Poisson alg. (by 1-3)

A classical mechanical system is a manifold \( P \) of a Poisson structure on \( C^\infty(P) \).

According to Heisenberg to quantize this system we replace the classical "algebra of observables" \( C^\infty(P) \) by an algebra of operators on a Hilbert space.

Dirac quantization prescription:

- functions \( f \) \( \mapsto \) operators \( \hat{f} \) — we require
  \[ \hat{f} \hat{g} - \hat{g} \hat{f} = i \hbar \{ f, g \} \]

Another operator, Planck's constant

We'd like to do this, but it's impossible!
fg = gf, for functions, but not operators \( \hat{f} \hat{g} \neq \hat{g} \hat{f} \).

We'll content ourselves w/ doing \( f \rightarrow \hat{f} \) when \( f \) is a linear function on \( P \). \( P \) was a manifold, now is a vector space.

We'll concentrate on \( \frac{q}{p} \rightarrow \hat{q} \) operators on some Hilbert space.

Since \( \{p, q\} = 1 \) then by Dirac \( [\hat{p}, \hat{q}] = -i \hbar \hat{I} \).

Schrödinger representation: \( \left( \mathcal{O} \rightarrow \text{End}(\mathcal{H}) \right) \)

\( \hat{q} : \mathcal{F} \rightarrow (\hat{q} \psi)(x) = x \psi(x) \)

\( (\hat{p} \psi)(x) = -i \hbar \frac{d\psi}{dx} \)

\( \hat{p} \) and \( \hat{q} \) are densely defined unbounded operators on \( \mathcal{H} \).

ex) \( C^\infty_0(\mathbb{R}) = \{ \text{compactly supported } C^\infty \text{ functions} \} \)

\( \overline{C^\infty_0(\mathbb{R})} = L^2(\mathbb{R}) \)

\( [\hat{p}, \hat{q}] = -i \hbar \hat{I} \) wants to eat a function...

\( [\hat{p}, \hat{q}] \psi(x) = \hat{p} \hat{q} \psi(x) - \hat{q} \hat{p} \psi(x) = \hat{p} x \psi(x) + \hat{q} i \hbar \frac{d\psi(x)}{dx} \)

\( = -i \hbar \frac{d}{dx} (x \psi(x)) + x i \hbar \frac{d\psi(x)}{dx} \)

\( = -i \hbar \psi(x) \)

\( = -i \hbar (\hat{I} \psi)(x) \)
operators:

- densely defined \( \tilde{d} \) bounded \( \Rightarrow \) extends by continuity to everywhere defined

- everywhere defined \( \tilde{\epsilon} \); unbounded requires nonconstructive method

\((P, C^\infty(P), \{ \cdot, \cdot \}) \mapsto (P \text{ is a } \mathcal{V} \text{ space}, P^* \text{ is a space of linear functions, } \quad W \text{ is a symplectic bilinear form on } P^*)\)

\[ W : P^* \times P^* \to \mathbb{IR} \]

linear functions \((f, g) \mapsto \{f, g\}\)

Poisson bracket of linear functions is a multiple of 1.
In general, \(\{\cdot, \cdot\}\) of 2 functions is a funct. But if they're linear \(\tilde{d}\) we assume \(\{p, q\} = 1\) we get the \(\{\cdot, \cdot\} \in \mathbb{IR}\).

A symplectic form on a \(\mathcal{V}\) space \(V\) is a skew-symmetric, non-degenerate bilinear form on \(V\).

\[ w(x, y) + w(y, x) = 0 \quad \forall x, y \in V \quad \text{(skew-symmetric)} \]
\[ w(x, y) = 0 \quad \forall y \in V \text{ then } x = 0. \quad \text{(non-degenerate)} \]
\[ w(ax + by, z) = aw(x, z) + bw(y, z) \quad \text{(bilinear form)} \]

\(\{p, q\} = 1\) implies \(\{p, f(q)\} = f'(q)\) if \(f\) is analytic.

A vector space w/ a symplectic form on it is called a symplectic vector space.
If \((V, w)\) is a symplectic \(V\)-space, we define the Weyl alg on \((V, w)\):

\[
W(V, w) := T(V) / \langle x \otimes y - y \otimes x = i\hbar w(x, y) \rangle.
\]

This is the alg, generated by \(V\) modulo the Heisenberg relns.

\[
[x, q] = -i \hbar w(x, y) \mathbb{1}
\]

Compare to Clifford alg on \((V, g)\): If \(\dim(V) = n\),

\[
C(V, g) := T(V) / \langle x \otimes y + y \otimes x = 2g(x, y) \rangle
\]

But \(W(V, w)\) is infinite dim'ed even if \(\dim V = 2\),

Let \(\dim V = 2\), \(p, q\) form a basis for \(V\) so we get:

\[
1, p, q, p^2, pq, qp, q^2, \ldots
\]

\(pq = qp - i\hbar w(p, q)\), so \(pq, qp\) are linearly dependent.

So we get:

\[
1, p, q, p^2, qp, q^2, p^3, q^2p, p^2q, \ldots
\]

so, \(W(V, w)\) is iso as \(V\)-spaces.

The space of polys on \((p, q)\) w/ a "twisted" product \(qp \neq pq\),

(we put all \(q\)'s in front of all \(p\)'s).
Thm: If \( \hat{x}, \hat{y} \) are elements of a Banach algebra then

\[
[\hat{x}, \hat{y}] \neq \hat{1}
\]

(we can't put a finite norm on a representation of the Heisenberg reals)

A Banach algebra is an algebra w/ a norm \( \| \cdot \| \) defined on it s.t. \( \| ab \| \leq \| a \| \| b \| \) and it's complete w/ \( \| \cdot \| \) norm topology.

pf: Assume \( [\hat{x}, \hat{y}] = \hat{1} \), then \( [x, y^n] = n y^{n-1} e_i y^n = 0 \) by induction (\( x \) acts like a deriv. of functs of \( y \))

\[
n y^{n-1} = xy^n - y^n x,
\]

\[
n \| y^{n-1} \| = \| xy^n - y^n x \| \leq \| xy^n \| + \| y^n x \| = 2 \| x \| \| y \| \| y^{n-1} \|
\]

so we can divide through by \( \| y^{n-1} \| \) and obtain:

\[
n \leq 2 \| x \| \| y \| \quad \forall n \to \infty
\]

We can fix this by considering \( A \mapsto e^{iA} \), which maps possibly unbounded self-adjoint operators to unitary operators (w/ norm = 1)

We had \( V = \mathbb{R}^2 \Rightarrow (p, q) \)

\[
(p \; \hat{\psi})(x) = x \psi(x), \quad (\hat{p} \; \psi)(x) = -i \hbar \frac{d}{dx} \psi(x) \quad \text{on } L^2(\mathbb{R}^2)
\]

\[
[e^{ikx} \; \hat{\psi}](x) = e^{ikx} \psi(x),
\]

\[
[e^{i\alpha \hat{p}} \; \hat{\psi}](x) = \psi(x - \alpha)
\]

\*

The exponential of the derivatives is translation by Taylor's Thm.
Let \( \varphi: V \rightarrow W(V, w) \)
\[ x \mapsto x \quad \text{(inclusion)} \quad \text{(linear)} \]

Let \( W: V \rightarrow \mathcal{U}(H) \) — unitary operators on \( H \)
\[ x \mapsto e^{i\varphi(x)} \quad Kx \mapsto e^{iW(kx)} = e^{ik\varphi(x)} \]

(\#) What does \( \varphi(x) \varphi(y) = \varphi(y)\varphi(x) = -i\hbar w(x, y) \) imply for \( W(x) \) and \( W(y) \)?

Thm — (Baker-Campbell-Hausdorff): If \([x, y]\) commutes with \( x \) and \( y \), then \( e^xe^y = e^{x+y+\frac{i}{2}[x,y]} \)

If we apply this to \( e^{i\varphi(x)} \)...

we get
\[ W(x)W(y) = e^{i\varphi(x)} e^{i\varphi(y)} = e^{i\varphi(x+y) + \frac{i}{2}[\varphi(x), \varphi(y)]} \]

\( \varphi \) is linear (\[ e^{i\varphi(x+y)} = e^{i\varphi(x)} e^{i\varphi(y)} \] by (\#))

\[ = W(x+y) e^{ik\frac{h}{2} w(x, y)} \]

A Weyl system on \((V, w)\) is a map \( W: V \rightarrow \mathcal{U}(H) \) such that \( W(x)W(y) = W(x+y) e^{ik\frac{h}{2} w(x, y)} \).

Fock Space

In 1-dim we had \( \hat{p}, \hat{q}, [\hat{p}, \hat{q}] = -i\hbar \)
\[ \hat{a} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{\hbar}} \hat{q} + \frac{i}{\hbar} \hat{p} \right) \]
\( \hat{\alpha} \) has an adjoint
\[
\hat{\alpha}^* = \frac{1}{\sqrt{2}} \left( \frac{i}{\lambda} \hat{\phi} - \frac{i}{\lambda} \hat{\rho} \right)
\]
chosen so that \( [\hat{\alpha}, \hat{\alpha}^*] = 1 \).

\( \hat{N} = \hat{\alpha}^* \hat{\alpha} \) is positive, self-adjoint and its spectrum is \( \mathbb{N} \).

\( \hat{N} \) has eigenvectors, one of them is zero: \( |0\rangle \) and satisfies
\[
\hat{N} |0\rangle = 0
\]
\( \hat{\alpha} |0\rangle = 0 \rightarrow \hat{\alpha} \) has one eigenvector \( \forall z \in \mathbb{C} \)
\[
\hat{\alpha} |z\rangle = z |z\rangle
\]

\[
\lambda \int |z\rangle \langle z| \, dz \, d\bar{z} \propto 1
\]

\( \forall z, w \in \mathbb{C} \)

For \( v \in \mathcal{H} \), \( e \) is a vector of norm 1, then
\[
|e\rangle \langle e, v\rangle \text{ is the projection of } v \text{ on the line spanned by } e.
\]

If \( |\psi\rangle \) is any state in \( \mathcal{H} \), then \( \psi(z) = \langle z, \psi\rangle \)

is a "wave function." Every state in \( \mathcal{H} \) gets represented

as a function from \( \mathbb{C} \) to \( \mathbb{C} \).

\[
\langle \psi, \phi \rangle = \lambda \int |\psi(z), \phi\rangle \langle z, \phi\rangle \, dz \, d\bar{z}
\]

\( = \lambda \int \hat{\psi}^*(z) \phi(z) \, dz \, d\bar{z} \)