5/19/03

Heisenberg's Theory of the Strong Interaction:

- $G = SU(2)$
- nucleons H_N = {massive spin-1/2 particles} $\otimes \mathbb{C}^2$

  ↖ defining rep of $SU(2)$. $SU(2)$ acts on $\mathbb{C}^2$ by mult. of matrices

H_π = {massive spin-0} $\otimes \mathfrak{sl}(2,\mathbb{C})$

$\cong \mathbb{C}^3$ ↖ complexified adj. rep.

$SU(2)$ acts on $\mathfrak{su}(2)$ by conjugation:

$$g: x \longmapsto gxg^{-1}$$

This is called the adjoint rep.

$\mathfrak{su}(2)$ is a 3-dim'l real Lie algebra, $\mathfrak{su}(2)\otimes\mathbb{C}$ is a 3-dim'l complex Lie alg.

* In general, $\mathfrak{su}(n)\otimes\mathbb{C} \cong \mathfrak{sl}(n,\mathbb{C})$ ← traceless $n\times n$ complex matrices

check: If $X \in \mathfrak{sl}(n)$ then $X = A + iB$ w/ $A, B \in \mathfrak{su}(n)$.

(This reminds us of how we can write any complex # as sum of real & complex part.)

This shows they're iso as v. spaces

Note: $X^* = A^* - iB^*$ (if $X = A + iB$)

$$= -A + iB$$

$\mathfrak{sl}(2,\mathbb{C})$ likes to act on $\mathbb{C}^2$

pions — nucleons

so: $A = \frac{1}{2}(X - X^*)$, $B = \frac{1}{2i}(X + X^*)$

Check this works:

1) A & B really are skew-adjoint
2) A & B really are traceless

$$\mathrm{tr}\, A = \frac{1}{2}[\mathrm{tr}(X) - \mathrm{tr}(X^*)] = \frac{1}{2}\left(\mathrm{tr}(X) - \overline{\mathrm{tr}(X)}\right) = 0$$

since $X \in \mathfrak{sl}(n)$ so $\mathrm{tr}(X) = 0$.

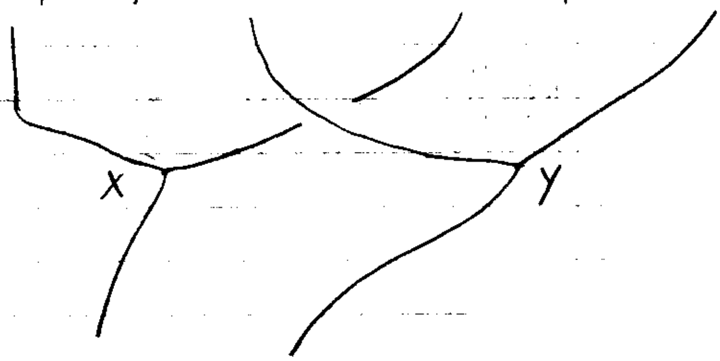
Also note: $\mathfrak{su}(n) \otimes \mathbb{C} \cong \mathfrak{sl}(n,\mathbb{C})$ as <u>Lie</u> <u>algebras</u>.

$SU(n)$ acts on $\mathfrak{sl}(n,\mathbb{C})$ by conjugation
$$g : X \longmapsto gXg^{-1},$$
and this is the complexified adjoint rep.

We also need interactions:

- $H_\pi \otimes H_N \longrightarrow H_N$ is given by:

$$H_\pi \otimes H_N = \{\text{massive spin-}0\} \otimes \mathfrak{sl}(2,\mathbb{C}) \otimes \{\text{massive spin-}1/2\} \otimes \mathbb{C}^2$$



$$H_N = \{\text{massive spin-}1/2\} \otimes \mathbb{C}^2$$

Note – pions hold nucleus together,
gluons hold proton (quarks) together (itself!)

Diagram on prev. pg is built from the intertwiners:

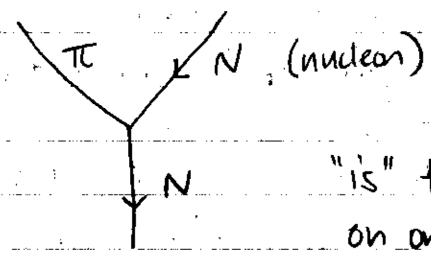
X: {massive spin-0} $\otimes$ {massive spin-$\frac{1}{2}$} $\longrightarrow$ {massive spin-$\frac{1}{2}$}

$Y: \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

Y is just action of $\mathfrak{sl}(2, \mathbb{C})$ on its defining rep $\mathbb{C}^2$.

Way pions act is same as way Lie alg. acts on a rep of it.
Note: Pions hold nucleus together.

In a picture:



"is" the action of a Lie alg. on one of its representations.

In general, "force-carrying particles" (like the pion) correspond to adjoint reps of internal symmetry grps i.e. to Lie algebras. (So a pion is a Lie alg. element!)

Recall: nucleon $\sim \mathbb{C}^2$ w/ basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = p$ proton

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = n$ neutron

pion $\sim \mathfrak{sl}(2, \mathbb{C})$ w/ basis $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \pi^+$ Note: $\pi^+ n = p$.
i.e. $Y(\pi^+ \otimes n) = p$.

So – charge is conserved.

positive charge, so takes something of no charge to something post. charged.

pion $\sim sl(2,\mathbb{C})$ basis: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \pi^+$ w/ $\pi^+ n = p$
charge conserved

neutral (no charge) $\rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \pi^0$ $\pi^0 p = p$, $\pi^0 n = -n$

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \pi^-$ w/ $\pi^- p = n$

neg. charged takes something of post. charge to something w/ no charge!

Note – we could have $\pi^0 p = \begin{pmatrix} i \\ 0 \end{pmatrix}$ which is a proton in a different state!

These bases are only correct up to normalization.

Note: This theory is incorrect because proton & neutron has different charges, π^+, π^0, π^- have different charges. But the charges only differ by 1.

One problem w/ Heisenberg's theory is that p & n and π^+, π^0, π^- have different charges. But they follow a simple pattern – they go in steps of 1.

Self-adjoint operators on Hilbert spaces correspond to observables and if
$$A\psi = \lambda\psi, \quad \lambda \in \mathbb{R}$$
then we say observable A takes the value λ when measured in state ψ (unit vector in Hilbert space).

Everything we have is an action of $sl(2,\mathbb{C})$.
We want a self-adjoint element of $sl(2,\mathbb{C})$ and to be an observable.

SU(2) also tells spin (ang. momentum) so σ_3 tells us ang. mom in z-axis.

In our problem all our Hilbert spaces are reps of $sl(2,\mathbb{C})$ so to find observables you can look for self-adjoint elements of $sl(2,\mathbb{C})$. Any real linear comb. of Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ will do! We'll try $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Actually, we'll try $\frac{\sigma_3}{2}$.

This gives an observable called $\underline{I_3}$, called the <u>third component of isospin</u>.

Let's work out the eigenvalue of I_3 for various particles, and hope it's related to their electric charge.

$$I_3 p = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}p.$$

So, a proton has $I_3 = 1/2$.

$$I_3 n = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1/2\, n,$$

So a proton has $I_3 = -1/2$.

Pions live in the complexified adj. rep., so $sl(2,\mathbb{C})$ acts by $[\cdot,\cdot]$ (or commutator) (what we get when differentiate conjugation).

$$[I_3, \pi^+] = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \pi^+$$

So, π^+ has $I_3 = 1$.

$[I_3, \pi^0] = 0$, so π^+ has $I_3 = 0$.

$[I_3, \pi^-] = -\pi^-$ so π^- has $I_3 = -1$.

We get:

		particle	I_3	electric charge (Q)
nucleon	N	p	1/2	1
		n	-1/2	0
pion	π	π^+	1	1
		π^0	0	0
		π^-	-1	-1

So, $I_3 = Q + \text{constant}$ where the constant depends on the irrep:
it's -1/2 for nucleon (p & n)
0 for pion

So physicists defined a quantity called hyperchange, Y, by:

$$Q = I_3 + \frac{Y}{2}$$

So - nucleon has hypercharge 1, pion has hypercharge 0.
This formula lives on in Standard Model.

In fact - there are ways for the particles we've discussed:

$\gamma, e, p, n, \pi^+, \pi^0, \pi^-, \mu$

to decay into other ones. This is not explained by our theories so far.

A neutron, left by itself, will decay:

$$n \longrightarrow p + e + ??$$

Today ?? is called $\overline{\nu_e}$, the anti-electron-neutrino.

mass of electron: .511 MeV = 511,000 eV

$\bar{\nu}_e$ was postulated to save conservation of energy & ang. momentum.

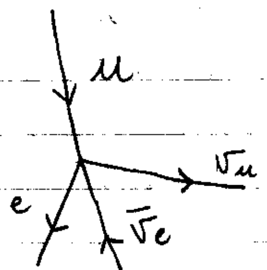
We now know that there is:

ν_e spin-½ charge-0 mass < 35 eV

$\bar{\nu}_e$ spin-½ charge-0 mass < 35 eV

The mean lifetime of a neutron is 918 seconds.

Also, the pions &, μ decay:



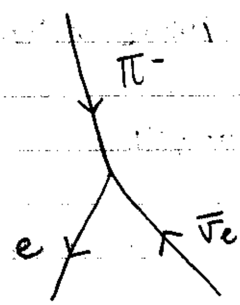
mean lifetime of μ: 2.2×10^{-6} seconds

where muon neutrino has:

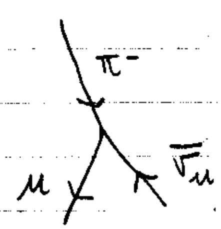
ν_μ spin-½ charge 0 mass < .2 MeV

Note: electron number (counting ν_e's) and muon number (counting ν_μ's) are conserved by these decays.

pion decay:

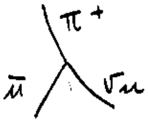


is possible, but rare!



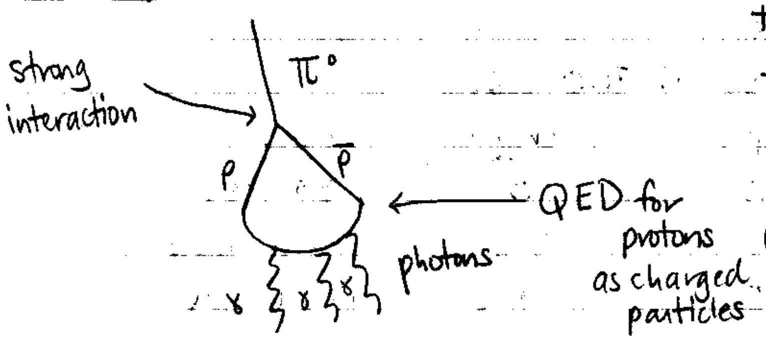
is what usually happens — mean lifetime is 2.6×10^{-8} sec.

Note: π^+ decay is same as for π^-, but w/ all particles replaced by its antiparticle:



pion decay:

things only decay into things lighter than it!



(quantum electro dynamics)

This is an old-fashioned explanation of how π^0 decays: picture is wrong but the output is correct — π^0 really does decay into photons.
Unlike the other decays, this doesn't require new physics beyond QED for protons and Heisenberg's theory.
It's also much faster: the mean lifetime of π^0 is 8.28×10^{-17} sec.

The "slow" decays involving neutrinos are explained by a third force — the weak force.

Rather than talking about this, we'll talk about the strong force some more —

baryon—heavy

It turns out that there are many particles interacting via the strong force. We call them "hadrons" (hadros = strong)
They're classified very roughly into:

- "mesons" (mesos — "middle") which have mass between mass of electron & proton
- "baryons" (baryos — "heavy") which have mass $\geq$ proton.

Particles other than hadrons & the photon were called "leptons" (light) e.g. e, ν_e, μ, ν_μ.

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Mesons w/ mass < 700 MeV

		spin	charge	mass
own anti-particle	π^0	0	0	135 MeV
	π^+] anti-particles	0	+1	139.6 MeV
(so is photon)	π^-	0	-1	139.6 MeV
	K^+] anti-particles	0	+1	493.7 MeV
	K^-	0	-1	493.7 MeV
	K^0	0	0	497.7 MeV
	$\overline{K^0}$	0	0	497.7 MeV
own antiparticle	η^0	0	0	548.8 MeV

Mesons — interact via strong force that are bosons — meaning have integer spins.

Classify these into "multiplets" - irreps of isospin $SU(2)$

e.g. $\pi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\pi^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\pi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

basis for $\mathfrak{sl}(2,\mathbb{C})$ — complexified adjoint rep.

one irrep of each dim 1, 2, 3, 4, ...

We saw $\underset{\text{charge}}{Q} = I_3 + \frac{Y}{2}$

where for π, $Y = 0$.

We group π's together since have roughly same mass.

Let's make the η^0 a multiplet all by itself.
So - this gives a 1-dim'l rep (since 1-particle)
so it's the trivial rep!

This is the trivial rep of $SU(2)$ on $\mathbb{C}$, so

$$\eta^\circ = 1 \in \mathbb{C}.$$

* In the n-dim'l rep of $SU(2)$, I_3 has eigenvalues $-\frac{(n-1)}{2}, -\frac{(n-1)}{2}+1, \ldots, \frac{n-1}{2}$ $\quad$ I_3 goes up in integer steps.

So, for pions, we have 3-dim'l rep w/ eigenvalues going from

$$-\frac{(3-1)}{2}, \; -\frac{(3-1)}{2}+1, \; \frac{3-1}{2} = -1, 0, 1$$

which is correct!

$$I_3 \pi^+ = 1, \quad I_3 \pi^\circ = 0, \quad I_3 \pi^- = -1$$

For eta: $I_3 = 0$ and $Q = 0$, so $Y = 0$.

We want to group the K's together. If we group all four of them together to get a 4 dim'l rep, their I_3's will increase in integer steps. But then $Q = I_3 + \frac{Y}{2}$ forces the charge to go up in integer steps. But they don't: $-1, 1, 0, 0$.

So we can't do that!

Instead we'll try two 2-dim'l irreps:

$K^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $K^\circ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{C}^2$ $\qquad I_3 K^+ = 1/2, \quad I_3 K^\circ = -1/2$

$\overline{K^\circ} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $K^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{C}^2$ $\qquad I_3 \overline{K^\circ} = 1/2 \quad I_3 K^- = -1/2$

$$Q = I_3 + \frac{Y}{2}$$

so for K^+, K^0, charge $= (1, 0)$

$I_3 = (1/2, -1/2)$

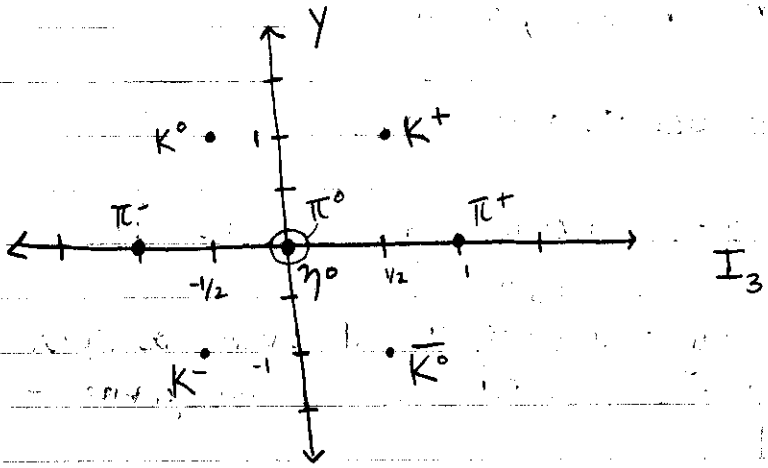
so $Y = (1, 1)$

for $\overline{K^0}, K^-$ charge $= (0, -1)$

$I_3 = (1/2, -1/2)$

so $Y = (-1, -1)$

Weight diagram of $\mathfrak{sl}(3, \mathbb{C})$:



This is the footprint of $A_2 = \mathfrak{sl}(3; \mathbb{C}) = \mathfrak{su}(3) \otimes \mathbb{C}$ complexified

Gell-Mann talked to R. Block about this q, Gell-Mann won Noble Prize

$I_3 \pi^+ = 1, \quad Q\pi^+ = 1 \quad \Rightarrow \quad Y\pi^+ = 0.$

Heisenberg: π^0, π^+, π^- basis of $\mathfrak{sl}(2, \mathbb{C})$

Gell-Mann: all 8 of these are a basis of some 8-dim'l Lie alg $= \mathfrak{sl}(3; \mathbb{C})$.

<u>Note</u>: $\mathfrak{sl}(3; \mathbb{C})$ – an adj. rep on itself.