5/27/03 **Jordan Algebras**

motivation – to formally capture aspects of observables in QM in an alg.

- Classical Mechanics – observables are real-valued functs.
- QM – observables are self-adjoint operators on some Hilbert space here – no nice mult. as for real-valued functs, since $(AB)^* \neq A^* B^*$ unless $AB = BA$
- However, we CAN raise the self-adj't operators to powers: $A^n = A \cdot \dots \cdot A$, so A^n is self-adjoint when A is.

Polarization suggests a new product:

$$2 \cdot a \circ b = (a+b)^2 - a^2 - b^2 = (ab + ba)$$

Defn: If A is an assoc. alg we can define a Jordan product on its underlying v. space $(A,+)$ by giving it a new product:

$$a \circ b = \frac{1}{2}(ab + ba).$$

Then, w/ this product $(A,+,\circ)$ is a Jordan algebra.

Note: A is an alg over some field k, $\mathrm{char}(k) \neq 2$.

Remarks: 1) Not every Jordan alg. arises this way! (i.e. come from an assoc. alg)
However – practically all Jordan algs. do arise this way. We call such Jordan algebras special.

2) The factor of "2" is optional. We get a Jordan alg either way, though not the same Jordan alg.

Assoc. Law: If all 3 inputs are same, we say power-associative, 2 inputs the same: alternative

$x(yx) = (xy)x$

Note; if we use the "2", we get $a \circ a = a^2$.

Thm: A Jordan product is commutative, but not associative.
* (Strange! We're used to losing commutativity before associativity!)

Also — a Jordan product is power-associative.

pf: Power-assoc: $a \circ \dots \circ a = a^n$
$$a(aa) = (aa)a \Rightarrow$$
$$a \circ (a \circ a) = (a \circ a) \circ a$$
need to use induction

is anything commut but not power-assoc?

Assoc: $(a \circ b) \circ c = \frac{1}{4}(abc + bac + cab + cba)$
$$a \circ (b \circ c) = \frac{1}{4}(abc + acb + bca + cba)$$
different unless A is commut.

ex of a power-assoc. law that doesn't come from commutativity:
$$a \circ ((a \circ a) \circ a) = (a \circ a) \circ (a \circ a)$$

Note: Jordan product has an identity.

$x^2 =$ product in assoc. alg $\neq$ product in Jordan alg.

Thm: A (special) Jordan algebra satisfies the Jordan identity:
$$(x^2 \circ y)x = x^2 \circ (y \circ x)$$

pf: $LHS = \frac{1}{4}(xxyx + yxxx + xxxy + xyxx)$

$RHS = \frac{1}{4}(xxxy + xxyx + xyxx + yxxx)$ ∎

* Check this holds when $x^2 = x \circ x$, but by note above — true!

At first these identities were enough:

<u>Defn</u>: A <u>Jordan alg</u> (no "special") is a v. space $(A,+)$ w/ a bilinear product $\circ$ satisfying commutativity and the Jordan identity, has a unit.
(Power-assoc. is implied.)
(aka a "Linear Jordan system") since by "algebra" people mean assoc.

A <u>special</u> Jordan alg. is one coming from an assoc. product in the way described. If not, we call it <u>exceptional</u>.

<u>Rmk</u>: The special Jordan algebras have extra identities, called s-identities. (infinitely many of them)
Given $x, y, z \in$ Jordan alg, we define bracket product:

$$\{xyz\} = (x \circ y) \circ z + (y \circ z) \circ x - (z \circ x) \circ y.$$

This is used in the <u>Glennie identity</u>:

$$2\{\{y\{xzx\}y\}z(x \circ y)\} - \{y\{x\{z(x \circ y)z\}x\}y\}$$

$$-2\{(x \circ y)z\{x\{yzy\}x\}\} + \{x\{y\{z(x \circ y)z\}y\}x\} = 0.$$

We can't derive this from the others.

<u>Defn</u>: We call a Jordan algebra J <u>formally real</u> if $a_j \in J$ and $a_1^2 + \dots + a_n^2 = 0 \Rightarrow a_j = 0 \ \forall j$.

You can do this for <u>any</u> algebra. These generalize real functions in classical mechanics.

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Jordan algebras: Finite dimensional formally real Jordan algebras are direct sums of simple ones.

h_n = $n \times n$ hermitian matrices over $\mathbb{K}$.

1) $h_n(\mathbb{R})$, $h_n(\mathbb{C})$, $h_n(\mathbb{H})$ $\quad a \cdot b = \frac{1}{2}(ab+ba)$

2) $h_3(\mathbb{O})$ (same product)

3) $\mathbb{R}^n \oplus \mathbb{R}$ w/ $(\vec{x},\alpha)\cdot(\vec{y},\beta) = (\alpha\vec{x}+\beta\vec{y}, \langle\vec{x},\vec{y}\rangle + \alpha\beta)$

the only non-special one.

$h_n(\mathbb{K})$ is in here.

<u>Thm</u>: If a v. space V w/ Lie bracket $[\cdot,\cdot]$ (antisymmetric, satisfying Jacobi id) and Jordan product $\circ$, then these came from an assoc. alg. V w/

$$[x,y] = \frac{1}{2}(xy-yx), \quad x\circ y = \frac{1}{2}(xy+yx)$$

iff (1) $(x\circ(y\circ z)) - ((x\circ y)\circ z) = [x,[y,z]] - [[x,y],z]$

(2) $[z\circ x, y] = [x, z\circ y] + [z, x\circ y] = [x,y]\circ z - [y,z]\circ x$.

<u>pf</u>: ($\Rightarrow$) Do it!

($\Leftarrow$) Define $xy = [x,y] + x\circ y$

Check $(xy)z - x(yz) \stackrel{?}{=} 0$

Eightfold Way

Recall: We grouped the 8 lightest mesons into irreps of isospin $SU(2)$

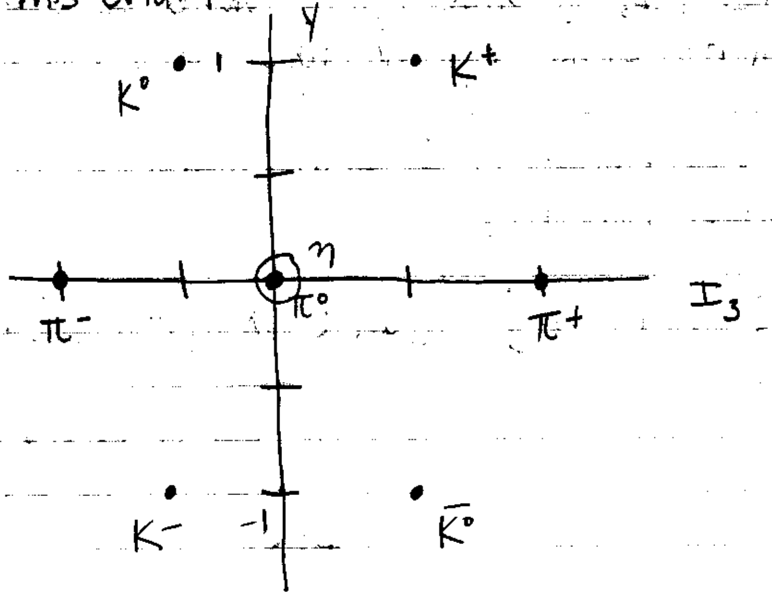
$Y = 0$ $\quad \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$ $\quad \begin{matrix} I_3 = 1 \\ I_3 = 0 \\ I_3 = -1 \end{matrix}$ $\qquad Q = I_3 + \frac{Y}{2}$

$Y = 1$ $\quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$ $\quad \begin{matrix} I_3 = 1/2 \\ I_3 = -1/2 \end{matrix}$

$Y = -1$ $\quad \begin{pmatrix} \overline{K^0} \\ K^- \end{pmatrix}$ $\quad \begin{matrix} I_3 = 1/2 \\ I_3 = -1/2 \end{matrix}$

$Y = 0$ $\quad (\eta)$ $\quad I_3 = 0$

and got this chart:



Now let's try to think of these 8 mesons as a basis for $\mathfrak{su}(3) \otimes \mathbb{C} = \mathfrak{sl}(3, \mathbb{C})$.
and find self-adjoint matrices

$$I_3, Y \in \mathfrak{sl}(3, \mathbb{C})$$

s.t. our 8 mesons are eigenvalues of the operators

$$[I_3, \cdot] \ \text{\&} \ [Y, \cdot]$$

whose eigenvalues are listed in the chart.

E.g. We know π^+ has $I_3 = 1$, $Y = 0$. So we want to find a 3x3 matrix $\pi^+ \in \mathfrak{sl}(3, \mathbb{C})$ w/

$$[I_3, \pi^+] = 1\pi^+$$
$$[Y, \pi^+] = 0\pi^+$$

Since $[I_3, \cdot]$ and $[Y, \cdot]$ have a basis of simultaneous eigenvectors, they must commute.

This will happen if I_3 and Y commute, by the Jacobi identity:

$$[[I_3, Y], X] = \underbrace{[I_3, [Y, X]] - [Y, [I_3, X]]}_{\text{If this is zero } \forall X, \text{ then so is the LHS.}}$$

In fact, this is an <u>iff</u> but let's pick I_3 & Y so they commute.

In Heisenberg's theory, we had $I_3 \in \mathfrak{sl}(2, \mathbb{C})$ equal to

$$I_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so now let's try

$$I_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$1 \neq -1 \neq 0$ no degeneracy of eigenvalues

This means Y has to be diagonal:

$$Y = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

and $Y \in \mathfrak{sl}(3, \mathbb{C}) \Rightarrow tr(Y) = a+b+c = 0$

Now let's find $\pi^+ \in \mathfrak{sl}(3, \mathbb{C})$ w/ $[I_3, \pi^+] = 1\,\pi^+$ and $[Y, \pi^+] = 0\,\pi^+$

In Heisenberg's $\mathfrak{sl}(2, \mathbb{C})$ theory:

$$\pi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Now we wants $\pi^+ \in \mathfrak{sl}(3, \mathbb{C})$ so try

$$\pi^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Check:

$$[I_3, \pi^+] = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = 1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

Now want

$$[Y, \pi^+] = \left[\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & a-b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= (a-b)\,\pi^+$$

This will be zero times π^+ if $a=b$, so

$$Y = \begin{pmatrix} a & & \\ & a & \\ & & -2a \end{pmatrix}$$

we can also try

$$\pi^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

As before we have

$$[I_3, \pi^-] = -\pi^-$$

and also

$$[Y, \pi^-] = \left[\begin{pmatrix} a & & \\ & a & \\ & & -2a \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} a-a & \end{pmatrix} = 0\,\pi^-$$

Finally

$$\pi^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{up to normalization}$$

has $[I_3, \pi^0] = 0$ since $\pi^0 = 2 I_3$

$$[Y, \pi^0] = \left[\begin{pmatrix} a & & \\ & a & \\ & & -2a \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \right] = 0$$ since they're diagonal

Now try a 3×3 matrix like

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left[I_3, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right] = \left[\tfrac{1}{2} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right] = \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So, if it works at all, it must be K^+ or $\overline{K^0}$.

$$\left[Y, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right] = \left[\begin{pmatrix} a & & \\ & a & \\ & & -2a \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3a & 0 \end{pmatrix}$$

$$= -3a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So if this is going to work, we need to have $-3a = \pm 1$ (Y of K^+ or $\overline{K^0}$).
In fact, either choice works just as well, so take the conventional choice:

$$\overline{K^0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$ and $-3a = -1$, so $a = 1/3$

So, $$Y = \begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{pmatrix}$$

Transposing gives antiparticles in Heisenberg's theory, so try

$$K^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Indeed –

$$[I_3, K^0] = -\tfrac{1}{2} K^0$$

$$[Y, K^0] = 1 K^0$$

as our chart says:

Now try: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Note:

$$\Bigg[\underbrace{\begin{pmatrix} 1/2 & & \\ & -1/2 & \\ & & 0 \end{pmatrix}}_{= I_3}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\Bigg] = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1/2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so must be a particle w/ $I_3 = 1/2$

$$\Bigg[\underbrace{\begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{pmatrix}}_{= Y}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\Bigg] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so, $Y = 1$ for this particle, so it's the K^+!

So, we guess $K^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\begin{aligned} [I_3, K^+]^{Tr} &= -[I_3^{\,tr}, K^{+\,tr}] \\ &= -[I_3, K^-] \end{aligned}$$

$[I_3, K^+]^{Tr} = (K^+)^{tr} = K^-$

diagonal

So, $[I_3, K^-] = -K^-$

i.e. transposing multiplies the eigenvalue of $[I_3, \cdot]$ by -1.

Similarly. for Y, so our guess $K^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is correct.

(it's the antiparticle of K^+)

What about the η? Need $\eta \in \mathfrak{sl}(3, \mathbb{C})$ st

$$[I_3, \eta] = 0\eta$$

$$[Y, \eta] = 0\eta$$

so $\eta = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$ and $\eta \in \mathfrak{sl}(3, \mathbb{C})$ implies $a + b + c = 0$

$$\eta = \begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix}$$

We also want η to be orthogonal to π^0.

(amplititude for finding a π^0 to be an η is zero (no time))

$$\langle \eta, \pi^0 \rangle = \left\langle \begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \right\rangle$$

$$:= \operatorname{tr}\left(\begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix}^* \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \right)$$

$$= \operatorname{tr}\left(\begin{pmatrix} \bar{a} & & \\ & \bar{b} & \\ & & -\bar{a}-\bar{b} \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \right)$$

$$= \bar{a} - \bar{b}$$

So we need $a = b$.

$$\eta = \begin{pmatrix} a & & \\ & a & \\ & & -2a \end{pmatrix}$$

Want an orthonormal basis, so

$$1 = \langle \eta, \eta \rangle = \operatorname{tr}\left(\begin{pmatrix} a & & \\ & a & \\ & & -2a \end{pmatrix}^* \begin{pmatrix} a & & \\ & a & \\ & & -2a \end{pmatrix} \right)$$

$$= \bar{a}a + \bar{a}a + 4\bar{a}a$$

$$= 6|a|^2$$

so we know $|a| = \frac{1}{\sqrt{6}}$ so by convention

we choose phase st.

$$\eta : \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$