6/2/03 Quarks & Standard Model

Gell-Mann saw that the 8 lightest mesons form a basis of

$$\mathfrak{sl}(3,\mathbb{C}) \cong \mathfrak{su}(3)\otimes\mathbb{C}$$

the complexified adj. rep of SU(3).

What about other particles (hadrons) & other irreps of SU(3)?

What about the defining rep of SU(3) on $\mathbb{C}^3$?

A 3-dim'l v. space so should have basis of 3 particles. But not corresponding to anything seen! They have charges that are non-integral.

Gell-Mann called them "<u>quarks</u>"

$$\underbrace{\underset{\text{up}}{u = \begin{pmatrix}1\\0\\0\end{pmatrix}} \qquad \underset{\text{down}}{d = \begin{pmatrix}0\\1\\0\end{pmatrix}}} \qquad \underset{\text{strange}}{s = \begin{pmatrix}0\\0\\1\end{pmatrix}}$$

has isospin 0 in this theory "eightfold way"

correspond to things w/ isospin up & isospin down

Not true in current theory.

$$I_3 = \begin{pmatrix}1/2 & 0 & 0\\ 0 & -1/2 & 0\\ 0 & 0 & 0\end{pmatrix}$$

so, up & down are eigenvectors w/ eigenvalues ±1/2 - called "isospin-up" and "isospin-down"

The strange quark shows up in "strange" particles — not occuring in Heisenberg's SU(2) theory i.e. Kaons.

The dual of the defining rep of SU(3) is a rep on $(\mathbb{C}^3)^*$ (iso. as a vector space but not as rep).

This has a dual basis of "antiquarks":
row vectors

$\bar{u} = (1\ 0\ 0)$ $\quad \bar{d} = (0\ 1\ 0)$ $\quad \bar{s} = (0\ 0\ 1)$

We can use quarks & antiquarks to build our 8 mesons, since:

$$\mathfrak{sl}(3, \mathbb{C}) \subseteq \mathbb{C}[3] \cong \mathbb{C}^3 \otimes (\mathbb{C}^3)^*$$

Given $f \in V^*$, $v \in V$, we get an elt $v \otimes f \in V \otimes V^*$ which gives a lin. transf. of V by:

$$(v \otimes f)(w) = f(w)\, v \in V \quad \forall w \in V.$$

v - quark, f - antiquark $\quad v \otimes f$ — meson

Note — $\mathfrak{sl}(3, \mathbb{C})$ has 8 dims.
$\mathbb{C}[3]$ has 9 — is there another particle?
YES!

For example — $\quad \pi^+ = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes (0\ 1\ 0)$
$= u \otimes \bar{d}$

In Heisenberg's theory, the positive pion was just upper left 2x2 block.

So a positive pion is an up quark & anti-(down quark)

Particles are elements of a vector space, so its OKAY that some mesons are linear combs of products of quark & antiquark.

Like, for example: $\pi^\circ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a "superposition" or "hybrid".

Mesons – are quark-anti-quark pairs
Baryons – built from 3 quarks.

2 quarks: $\mathbb{C}^3 \otimes \mathbb{C}^3 \cong S^2\mathbb{C}^3 \oplus \Lambda^2\mathbb{C}^3$ — $\dim = \frac{3 \cdot 2}{1 \cdot 2} = 3$

can't pick same quark twice

the rep isn't reducible anymore! is direct sum of reps

$\dim = \frac{3 \cdot 4}{1 \cdot 2} = 6$

can pick same quark twice

$\dim \mathbb{C}^3 \otimes \mathbb{C}^3 = 9$, so $\mathbb{C}^3 \otimes \mathbb{C}^3$ is a sum of 2 irreps.

3 quarks $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \cong \mathbb{C}^3 \otimes (S^2\mathbb{C}^3 \oplus \Lambda^2\mathbb{C}^3)$

$\Lambda^2\mathbb{C}^3 \cong \mathbb{C}^{3*}$

$$\dim = \quad \cong \underbrace{\overset{3}{\mathbb{C}^3} \otimes \overset{6}{S^2\mathbb{C}^3}}_{18} \oplus \underbrace{\mathbb{C}^3 \otimes \mathbb{C}^{3*}}_{9}$$

$$\cong S^3\mathbb{C}^3 \oplus \mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C}) \oplus \mathbb{C}$$

$\dim = \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} = 10$ $\dim = 8$ $\dim = 8$ $\dim = 1$ trivial rep

The more familiar baryons live in the "decuplet" $S^3\mathbb{C}^3$ (gang of 10 things) & the "octet" $\mathfrak{sl}(3,\mathbb{C})$ (the 1st one shown).

$S^3\mathbb{C}^3$ = homogeneous deg-3 polynomials in u, d, s.

Ex) $\Omega^- = sss \in S^3\mathbb{C}^3$ strange quark is heaviest,

So, Ω^- is the heaviest baryon in decuplet. Its properties were predicted by Gell-Mann before it was found, so won Nobel Prize.

6/3/03

Gell-Mann organized the hadron "zoo" into irreps of $SU(3)$ and postulated <u>quarks</u> as particles transforming in the defining rep of $SU(3)$ on $\mathbb{C}^3$, $SU(3)$ is symmetry group mixing up the u, d, s quarks.

quarks are supposed be to spin-1/2 particles
• can't have 2 in same state!
but we have 3 in same state.

<u>Problem</u>: The baryon decuplet, $S^3\mathbb{C}^3$ contains particles like

$$\Omega^- = sss \in S^3\mathbb{C}^3$$

We have 3 identical quarks w/ same spin in the same state — violating Pauli exclusion principle.

<u>Answer</u>: postulate that quarks come in 3 "colors" — red, green & blue and that
(flavors — up, down, strange)

$$\Omega^- = s_r s_g s_b$$

Introduce a new $SU(3)$ that mixes up the 3 colors of quarks, in addition to Gell-Mann's $SU(3)$ mixing up the 3 "flavors" u, d, s. The whole symmetry group is then

$$\underbrace{SU(3)_{\text{flavor}}} \times \underbrace{SU(3)_{\text{color}}}$$

approximate symmetry (eventually gets thrown out)

turns out to be fundamental — this is the one in the Standard Model $SU(3) \times SU(2) \times U(1)$.

u, d produce protons, neutrons

Problem - Kaons decayed much slower than predicted.

Answer - Glashow - Iliopolis - Maiani suggested a fourth quark: "charm." This cancelled out some of the amplitude to decay.

Now: $\overset{Y = 1/3}{\begin{pmatrix} u \\ d \end{pmatrix}}$ ← charge 2/3, charge -1/3 ↗ $\quad \overset{Y = -1}{\begin{pmatrix} \nu_e \\ e \end{pmatrix}}$ ← charge 0, ↖ charge -1 first generation

Isospin doublets

$\begin{pmatrix} c \\ s \end{pmatrix}$ ← charge 2/3, charge -1/3 $\quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}$ ← charge 0, ↖ charge -1 second generation

$$Q = I_3 + \frac{Y}{2}$$

Salam also invented this extra quark just because it looked pretty. Mysterious - why is there one kind of lepton for each quark?

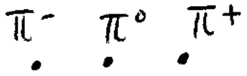
With 4 quarks, $SU(3)_{\text{flavor}}$ becomes $SU(4)_{\text{flavor}}$ q_i
hadrons should be
irreps of $SU(4)$.
Each new quark gives new mesons.
Mesons are now:

$$\begin{array}{ccccc} su(2)\otimes\mathbb{C} & \subseteq & su(3)\otimes\mathbb{C} & \subseteq & su(4)\otimes\mathbb{C} \\ \wr\wr & & \wr\wr & & \wr\wr \\ sl(2,\mathbb{C}) & \subseteq & sl(2,\mathbb{C}) & \subseteq & sl(4,\mathbb{C}) \\ \text{3-dim} & & \text{8-dim} & & \text{15-dim} \end{array}$$

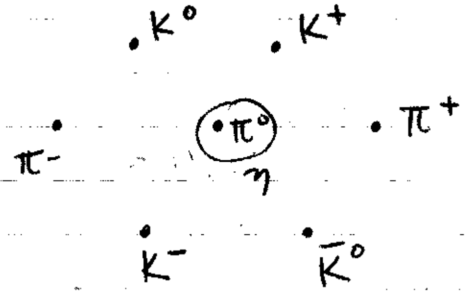
(since $sl(4,\mathbb{C})$ are 4x4 matrices (16 dim) w/ tr=0

So 15 dim. ↑

$\mathfrak{sl}(2,\mathbb{C})$:



$\mathfrak{sl}(3,\mathbb{C})$:



On handout: $\mathfrak{sl}(4,\mathbb{C})$ has 4 guys in center.

$$\mathbb{C}^4 \otimes \mathbb{C}^{4*} = \underset{\underset{\mathfrak{sl}(4,\mathbb{C}) \oplus \mathbb{C}}{\mathrel{\wr\Vert}}}{\mathbb{C}[4]} \qquad \text{16-dim}$$

└ multiples of id. matrix

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes (1000) + \dots + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \otimes (0001)$$

$$= u \otimes \bar{u} + d \otimes \bar{d} + s \otimes \bar{s} + c \otimes \bar{c}$$

$\dim \Lambda^p \mathbb{C}^n = \binom{n}{p}$

$\dim S^p \mathbb{C}^n = \binom{n+p-1}{p}$

Baryons now include:

$$S^3\mathbb{C}^2 \subseteq S^3\mathbb{C}^3 \subseteq S^3\mathbb{C}^4$$

$\dim = \frac{2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3} = 4$ $\qquad$ $\dim = \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} = 10$ $\qquad$ $\dim = \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20$

But $SU(4)_{\text{flavor}}$ is even more approximate than $SU(3)_{\text{flavor}}$.

Eventually, we give up on $SU(n)_{\text{flavor}}$ and go to...

the Standard Model :

$$\underbrace{SU(3)}_{\substack{\text{mixes up colors} \\ \text{of quarks}}} \times \underbrace{SU(2)}_{\substack{\text{mixes up} \\ \text{weak isospins} \\ \text{e.g. } u \,\&\, d \\ \nu_e \,\&\, e}} \times \underbrace{U(1)}_{\text{hypercharge}}$$

(Heisenberg – theory mixes up proton &, neutron)

(Gell-Mann – theory mixes up quarks, 3 things ... but really amounts to mixing 2 things since
proton = uud   neutron = ddu

Then we introduce force-carrying particles which live, as usual, in complexified adjoint rep of this group:

Strong Force – hold quarks together
: quarks stick together by transferring gluons

Strong force

$$\Big(\underset{\substack{\uparrow \\ 8 \text{ gluons} -\\ \text{hold quarks} \\ \text{together}}}{su(3)} \oplus \underset{\substack{\uparrow \\ W_1, W_2, W_3}}{su(2)} \oplus \underset{\substack{\nwarrow \\ W_0}}{u(1)}\Big) \otimes \mathbb{C}$$

or

$$W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$W^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$W_3 = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The formula $Q = I_3 + \frac{Y}{2}$ gives

$$\underset{\text{photon}}{\gamma} = W_3 + \frac{W_0}{2} \quad \text{(not normalized)}$$

The 2nd eqn on prev pg expresses the photon in terms of the W's. But there's also

$$Z^0 = -\frac{1}{2} W_3 + W_0$$

chosen to be orthogonal to W^+, W^-, and γ.

So we say $SU(2) \times U(1)$ is associated to the "electroweak force." We have

W^+, W^-	W bosons
Z^0	Z boson
γ	photon

as observed particles.

$su(2)$ w/ $[\cdot,\cdot]$ same as $\mathbb{R}^3$ w/ cross product has no preferred direction

Why is "I_3 direction" in $su(2)$ special? "Spontaneous symmetry breaking" due to Higgs boson. Let's look at some weak interactions.

$$\nu_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \qquad e = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$$

$$W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in sl(2, \mathbb{C})$$

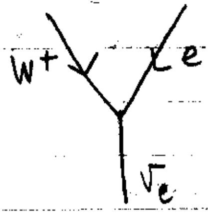
$$sl(2,\mathbb{C}) \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$T \otimes V \longrightarrow TV$$

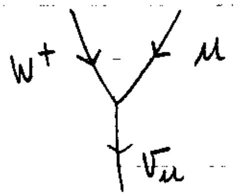
$$\underset{W^+}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \underset{e}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$W^+ \, e \longrightarrow \nu_e$$

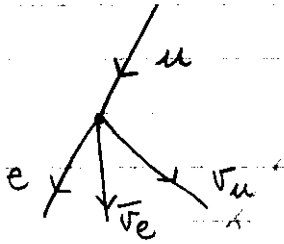
or



Also: same for 2nd generation:

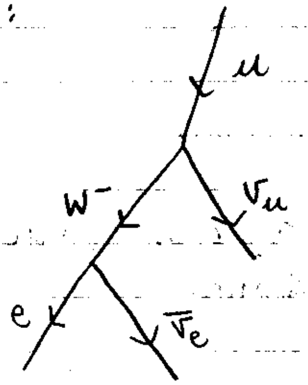


Sticking these together, we get muon decay:

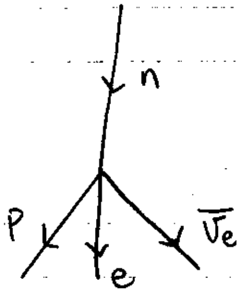


old picture

is really:



Or – neutron decay:



old picture

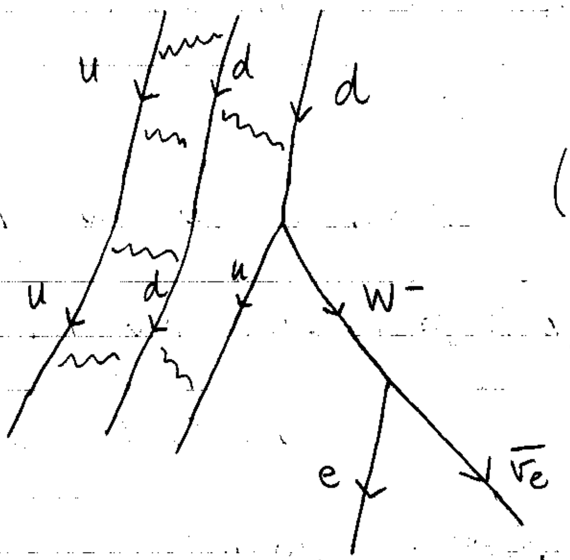
But n, p are baryons – made up out of quarks.

neutron

Really n is built from an up quark & 2 downs
$+\frac{2}{3}-\frac{1}{3}-\frac{1}{3}=0$ charge

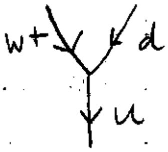
p is built from a down & 2 ups $-\frac{1}{3}+\frac{2}{3}+\frac{2}{3}=1$

quarks exchange gluons & colors



$W^+ \otimes d \rightarrow u$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



End of Standard Model discussion!

Futuristic Physics— which may not even be true!

There's a subgroup of $SU(3)\times SU(2)\times U(1)$ called N, which acts as identity on all particles in the Standard Model.
$N \cong \mathbb{Z}_6$.
So— <u>true</u> symmetry group is

$$SU(3)\times SU(2)\times U(1)\big/ N \cong S(U(3)\times U(2))$$

$$\cong \left\{ \begin{pmatrix} e^{-2i\theta}g & 0 \\ 0 & e^{3i\theta}h \end{pmatrix} \;\middle|\; \begin{matrix} g\in SU(3) \\ h\in SU(2) \\ e^{i\theta}\in U(1) \end{matrix} \right\}$$

$$\subseteq SU(5)$$

Any rep of true gauge grp: $SU(3) \times SU(2) \times U(1)$ gives a rep of quotient grp. (or vice versa)

We can try to lump together various irreps of $S(U(3) \times U(2))$ corresponding to different fermions, into irreps of $SU(5)$.
in a given generation

We'll think of reps of $SU(5)$ and see what reps of $S(U(3) \times U(2))$ they correspond to.

1) The trivial rep of $SU(5)$ on $\mathbb{C}$ restricts to the trivial rep of $S(U(3) \times U(2))$, or trivial rep of $SU(3) \times SU(2) \times U(1)$.
This rep corresponds to the right-handed electron neutrino: $\mathbb{C}, \mathbb{C}, \mathbb{C}_0$.
(This was the last particle to be found — similar to how zero is last natural # to be discovered, since electron neutrino acts trivially.)

2) The defining rep of $SU(5)$ on $\mathbb{C}^5$ restricts to $\mathbb{C}^3 \oplus \mathbb{C}^2$ as a rep of $S(U(3) \times U(2))$ or as a rep of $SU(3) \times SU(2) \times U(1)$:

$$\underbrace{\overset{\text{defining}}{\mathbb{C}^3} \otimes \overset{\text{trivial}}{\mathbb{C}} \otimes \mathbb{C}_{-2/3}}_{d^R \text{ right-handed down quark}} \oplus \underbrace{\overset{\text{triv.}}{\mathbb{C}} \otimes \overset{\text{defining}}{\mathbb{C}^2} \otimes \mathbb{C}_1}_{\text{anti-}(\nu_e^L, e^L) \text{ i.e. } (\bar{\nu}_e^R, \bar{e}^R)}$$

$$\begin{array}{cc} \begin{pmatrix} e^{-2i\theta} g & 0 \\ 0 & e^{3i\theta} h \end{pmatrix} & \begin{matrix} \square \\ \square \\ \square \\ \square \\ \square \end{matrix} \end{array}$$

if $\alpha = e^{i\theta}$ then $e^{-2i\theta}$ is how its acting

$\alpha^{-2} = e^{-2i\theta}$ so $y = -2/3$

if $\alpha = e^{i\theta}$, then acts as $e^{3i\theta}$

$$\alpha^3 = e^{3i\theta}$$
$$\overset{\shortparallel}{\alpha^{3y}} \quad \text{where } y = 1$$

Want a 10-dim'l rep of SU(5).

3) The rep of SU(5) on $\Lambda^2 \mathbb{C}^5$ has dim $\frac{5\cdot 4}{1\cdot 2} = 10$,

$(\mathbb{C} \oplus \mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5)$

This is good because $1 + 5 + 10 = 16 = \dim F$ where F is the "fermion rep".

Note: $\Lambda^2(V \oplus W) = \Lambda^2 V \oplus V \otimes W \oplus \Lambda^2 W$

gives

$$\Lambda^2\mathbb{C}^5 \cong \Lambda^2(\mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3} \oplus \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1)$$

$$\cong \Lambda^2(\mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3}) \oplus \overset{\overset{\mathbb{C}^3}{\wr\|}}{(\mathbb{C}^3 \otimes \mathbb{C})} \otimes \overset{\overset{\mathbb{C}^2}{\wr\|}}{(\mathbb{C} \otimes \mathbb{C}^2)}$$
$$\otimes \underset{\underset{\mathbb{C}_{1/3}}{}}{\underset{\wr\|}{(\mathbb{C}_{-2/3} \otimes \mathbb{C}_1)}}$$
$$\oplus \Lambda^2(\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1)$$

$$\cong \Lambda^2(\mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3}) \oplus \underbrace{\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{1/3}}_{(u^L, d^L)}$$
$$\oplus \Lambda^2(\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1)$$

[As reps of SU(n), $\Lambda^p \mathbb{C}^n = \Lambda^{n-p}(\mathbb{C}^{n*})$ so

$$\Lambda^2\mathbb{C}^3 = \Lambda^1 \mathbb{C}^{3*} = \mathbb{C}^{3*}$$

$$\cong \underbrace{\mathbb{C}^{3*} \otimes \mathbb{C} \otimes \mathbb{C}_{-4/3}}_{\text{anti-}(u^R) = \bar{u}^L} \oplus \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{1/3}$$

$\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}_2$ > anti-$(e^R) = \bar{e}^L$

$$\Lambda^2\mathbb{C}^2 = \Lambda^0 \mathbb{C}^{2*} = \mathbb{C}$$

↑ 0-forms - like functions

$(\Lambda^p \mathbb{C}^n)^* \cong \Lambda^{n-p} \mathbb{C}^n$ dual of p-form is n-p form

Every fermion in F or its antiparticle shows up, once, in

$$X = \mathbb{C} \oplus \mathbb{C}^5 \oplus \Lambda^2 \mathbb{C}^5$$

so fermions & their antiparticles are:

$$F \oplus F^* = X \oplus X^* \quad \text{though } F \not\cong X.$$

$$X \oplus X^* = \underset{\Lambda^0\mathbb{C}^5}{\underset{\shortparallel}{\mathbb{C}}} \oplus \underset{\Lambda^1\mathbb{C}^5}{\underset{\shortparallel}{\mathbb{C}^5}} \oplus \Lambda^2 \mathbb{C}^5 \oplus \Lambda^3 \mathbb{C}^5 \oplus \Lambda^4 \mathbb{C}^5 \oplus \Lambda^5 \mathbb{C}^5$$

$$= \Lambda \mathbb{C}^5$$

Moral: Fermions & antifermions form $\Lambda \mathbb{C}^5$ as a rep of $S(U(3) \times U(2))$.