Gell-Mann saw that the 8 lightest mesons form a basis of
\[ \mathfrak{su}(3) \otimes \mathbb{C} \]
the complexified adj. rep of \( \mathfrak{su}(3) \).

What about other particles (hadrons) or other irreps of \( \mathfrak{su}(3) \)?
What about the defining rep of \( \mathfrak{su}(3) \) on \( \mathbb{C}^3 \)?
A 3-dim'le v. space so should have basis of 3 particles.
But not corresponding to anything seen! They have charges that are non-integral.
Gell-Mann called them "quarks".

\[
\begin{align*}
\mathbf{u} & = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \mathbf{d} & = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \mathbf{s} & = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{align*}
\]

\( \mathbf{u} \) \( \mathbf{d} \) \( \mathbf{s} \) has isospin 0 in this theory "eightfold way"

\( \mathbf{u} \) \( \mathbf{d} \) \( \mathbf{s} \) correspond to things w/ isospin up \( \mathbf{u} \), isospin down \( \mathbf{d} \), isospin down \( \mathbf{s} \)

\( I_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \) so, \( \mathbf{u}, \mathbf{d}, \mathbf{s} \) are eigenvectors w/ eigenvalues \( \pm \frac{1}{2} \) - called "isospin-up" and "isospin-down"

The strange quark shows up in "strange" particles - not occurring in Heisenberg's \( \mathfrak{su}(2) \) theory, i.e. kaons.

The dual of the defining rep of \( \mathfrak{su}(3) \) is a rep on \( (\mathbb{C}^3)^* \) (iso. as a vector space but not as rep).
This has a dual basis of "antiquarks:
row vectors

\[ \bar{u} = (1 \ 0 \ 0) \quad \bar{d} = (0 \ 1 \ 0) \quad \bar{s} = (0 \ 0 \ 1) \]

We can use quarks \( q \), antiquarks to build our
8 mesons, since:

\[ \mathcal{M}(3, \mathbb{C}) \cong \mathbb{C}[3] \cong \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \]

Given \( f \in V^* \), \( v \in V \), we get an elt \( v \otimes f \in V \otimes V^* \)
which gives a lin. transf. of \( V \) by:

\[ (v \otimes f)(w) = f(w)v \quad \forall \ v \in V, \ \forall \ w \in \mathcal{V} \]

\( V \)-quark, \( f \)-antiquark \( v \otimes f \)-meson

Note — \( \mathcal{M}(3, \mathbb{C}) \) has 8 dims.
\( \mathbb{C}[3] \) has 9 — is there another particle?

YES!

For example — \( \Pi^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} \)

In Heisenberg's theory, the positive pion was just upper left
2x2 block.

So a positive pion is an up quark \( q \), anti-(down quark)

Particles are elements of a vector space, so its
OKAY that some mesons are linear combs
of products of quark \( q \), antiquark.
Like, for example: \( \Pi^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) is a "superposition" or "hybrid".

Mesons - are quark-anti-quark pairs

Baryons - built from 3 quarks.

2 quarks: \( C^3 \otimes C^3 = S^2 C^3 \oplus \Lambda^2 C^3 \quad \text{dim} = \frac{3 \cdot 2}{1 \cdot 2} = 3 \)

(\text{the rep isn't reducible anymore} \text{ is direct sum of reps})

\( \text{can't pick same quark twice} \)

\( \text{can pick same quark twice} \)

\( \text{dim} C^3 \oplus C^3 = 9, \text{ so } C^3 \oplus C^3 \text{ is a sum of 2 irreps.} \)

3 quarks \( C^3 \otimes C^3 \otimes C^3 = C^3 \otimes (S^2 C^3 \oplus \Lambda^2 C^3) \)

\( \text{dim} = \frac{3 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} = 35 \)

\( \text{dim} S^2 C^3 = 6 \)

\( \text{dim} \Lambda^2 C^3 = 3 \)

\( \text{dim} C^3 = 9 \)

\( \text{dim} \otimes = 3 \cdot 4 \cdot 5 = 60 \)

\( \text{dim} \oplus = 10 \)

\( \text{dim} \otimes = 8 \)

\( \text{dim} \oplus = 8 \)

\( \text{dim} \Lambda = 1 \)

Trivial rep

The more familiar baryons live in the:

"decuplet" \( S^3 C^3 \) (gang of 10 things)

"octet" \( \mathfrak{sl}(3, C) \) (the 1st one shown)

\( S^3 C^3 = \text{homogeneous deg-3 polynomials in } u, d, s \)

Ex) \( \Lambda^- = s s s \in S^3 C^3 \)

Strange quark is heaviest.

So, \( \Lambda^- \) is the heaviest baryon in decuplet. Its properties were predicted by Gell-Mann before it was found, so won Nobel Prize.
Gell-Mann organized the hadron "zoo" into irreps of SU(3) and postulated quarks as particles transforming in the defining rep of SU(3) on $C^3$. SU(3) is symmetry group mixing up the u, d, s quarks.

Problem: The baryon decuplet, $S^3C^3$ contains particles like $\Lambda^- = sss \in S^3C^3$.

We have 3 identical quarks w/ same spin in the same state — violating Pauli exclusion principle.

Answer: postulate that quarks come in 3 "colors"— red, green & blue and that (flavors— up, down, strange) $\Lambda^- = S_r S_g S_b$.

Introduce a new SU(3) that mixes up the 3 colors of quarks, in addition to Gell-Mann's SU(3) mixing up the 3 "flavors" u, d, s. The whole symmetry group is then

$$SU(3)_{\text{flavor}} \times SU(3)_{\text{color}}$$

approximate symmetry (eventually gets thrown out)

turns out to be fundamental — this is the one in the Standard Model

$$SU(3) \times SU(2) \times U(1)$$
Problem - Kaons decayed much slower than predicted.

Answer - Glashow - Iliopolis - Maiani suggested a fourth quark: "charm." This cancelled out some of the amplitude to decay.

Now:

\[
\begin{align*}
\begin{pmatrix} u \\ d \end{pmatrix} & \quad \text{charge } \frac{2}{3} & & \begin{pmatrix} u_e \\ e \end{pmatrix} & \quad \text{charge } -1 \\
\begin{pmatrix} c \\ s \end{pmatrix} & \quad \text{charge } \frac{1}{3} & & \begin{pmatrix} \nu_u \\ \mu \end{pmatrix} & \quad \text{charge } 0
\end{align*}
\]

\( Q = I_3 + \frac{y}{2} \)

Salam also invented this extra quark just because it looked pretty. Mysterious - why is there one kind of lepton for each quark?

With 4 quarks, \( \text{SU}(3) \) flavor becomes \( \text{SU}(4) \) flavor.

Hadrons should be irreps of \( \text{SU}(4) \).

Each new quark gives new mesons.

Mesons are now:

\[
\text{su}(2) \otimes \mathbb{C} \leq \text{su}(3) \otimes \mathbb{C} \leq \text{su}(4) \otimes \mathbb{C}
\]

\[
\begin{align*}
\text{SS} & \quad \text{SS} & \quad \text{SS} \\
\text{sl}(2, \mathbb{C}) & \leq \text{sl}(2, \mathbb{C}) & \leq \text{sl}(4, \mathbb{C})
\end{align*}
\]

- 3-dim
- 8-dim
- 15-dim

(since \( \text{sl}(4, \mathbb{C}) \) are 4x4 matrices (16 dim) w/ \( tr = 0 \))
\[ \text{On handout: } \Lambda(4, \mathbb{C}) \text{ has 4 guys in center.} \]

\[ \begin{align*}
\Lambda^c & \cong \mathbb{C}[4] \quad 16 \text{-dim} \\
\Lambda^c & \cong \mathbb{C} \quad \text{multiples of id. matrix}
\end{align*} \]

\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes (1000) + \ldots + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes (0001) \]

\[ \dim \Lambda^c = \binom{n+p-1}{p} \]

\[ \dim S^c = \binom{n+p-1}{p} \]

\[ \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix} = 4 \]

\[ \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{pmatrix} = 10 \]

\[ \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} = 20 \]

**Baryons now include:**

\[ S^3 C^2 \subseteq S^3 C^3 \subseteq S^3 C^4 \]

**But Su(4)_{flavor} is even more approximate than Su(3)_{flavor}**.

Eventually, we give up on Su(n)_{flavor} and go to...
the Standard Model:

\[ \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \]

mixes up colors of quarks
weak isospins
  e.g. \( u, d, d \)
\( e, e, e \)

(Heisenberg - theory mixes up proton & neutron)
(Gell-Mann - theory mixes up quarks, 3 things... but really amounts to mixing 2 things since proton = \( uud \)  neutron = \( ddu \)

Then we introduce force-carrying particles which live, as usual, in complexified adjoint rep of this group:

**Strong Force** - hold quarks together
quarks stick together by transferring gluons

\[ \text{SU}(3) \oplus \text{SU}(2) \oplus \text{U}(1) \] \( \otimes \) \( \mathbb{C} \)

\[ \uparrow \quad W_1, W_2, W_3 \quad \uparrow \quad W_0 \]

8 gluons -
or
hold quarks together
\[ W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]
\[ W^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]
\[ W^3 = Y_2 \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \]

The formula
\[ Q = I_3 + \frac{Y}{2} \]
gives
\[ Y = W_3 + \frac{W_0}{2} \] (not normalized)

\[ \text{photon} \]
The 2nd eqn on prev pg expresses the photon in terms of the W's. But there's also

\[ Z^0 = -\frac{1}{2} W_3 + W_0 \]

chosen to be orthogonal to \( W^+, W^-, \) and \( \delta \).

So we say \( SU(2) \times U(1) \) is associated to the "electroweak force." We have

- \( W^+, W^- \) W bosons
- \( Z^0 \) Z boson
- \( \delta \) photon

\( SU(2) \times U(1) \) as observed particles.

Why is "\( I_3 \) direction" in \( SU(2) \) special?
"Spontaneous symmetry breaking" due to Higgs boson. Let's look at some weak interactions.

\[ \nu_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \quad e = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2 \]

\[ W^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in SL(2, \mathbb{C}) \]

\[ SL(2, \mathbb{C}) \otimes \mathbb{C}^2 \rightarrow \mathbb{C}^2 \]

\[ T \otimes V \rightarrow TV \]

\[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] or \( W^+ e \rightarrow \nu_e \)
Also, same for 2nd generation:

\[ w^+ \rightarrow \mu \] 
\[ \nu_{\mu} \]

Sticking these together, we get muon decay:

\[ \mu \]
\[ e \]
\[ \nu_e \]
\[ \bar{\nu}_e \]

is really:

\[ w^- \]
\[ \nu_{\mu} \]
\[ e \]
\[ \bar{\nu}_e \]

or- neutron decay:

\[ n \]
\[ p \]
\[ \nu_e \]
\[ \bar{\nu}_e \]

But \( n, p \) are baryons - made up out of quarks.
Really, n is built from an up quark, 2 downs
\[ +\frac{2}{3} - \frac{1}{3} - \frac{1}{3} = 0 \] charge

p is built from a down, 2 ups
\[ -\frac{1}{3} + \frac{2}{3} + \frac{2}{3} = 1 \]

end of standard model discussion.

Futuristic physics—there's a subgroup of $SU(3) \times SU(2) \times U(1)$ called $N$, which acts as identity on all particles in the standard model. $N \cong \mathbb{Z}_6$.

So the true symmetry group is $SU(3) \times SU(2) \times U(1) / N \cong S(U(3) \times U(2))$.

\[ \cong \left\{ \begin{bmatrix} e^{-2i\theta} & 0 \\ 0 & e^{3i\theta} \end{bmatrix} \right\}, \begin{bmatrix} g \in SU(3) \\ h \in SU(2) \\ e^i \in U(1) \end{bmatrix} \]

\[ \subseteq SU(5) \]
The defining rep of $SU(5)$ on $C^5$ restricts to $C^3 = \mathbb{C} \oplus \mathbb{C}^2$ as a rep of $SU(3)_c \times SU(2)_w$ or as a trivial rep of $SU(3)_c$.

This rep corresponds to the right-handed electron (This was the last particle to be found — similar to how zero is last natural # to be discovered, since electron neutrino acts trivially)

we can try to lump together various irreps of $SU(5)$ corresponding to different fermions into irreps of $SU(3)_c \times SU(2)_w$ gives a rep of quotient group (or via usual)

Any rep of $SU(2)_w \times SU(3)_c$ gives

As $\rho \equiv e^{i\phi} \rho$ where $\frac{\phi}{\pi} = \frac{\pi}{3}$, then acts as $e^{i\phi}$

so $\rho = e^{-i\phi}$ is how its acting
Want a 10-dim'l rep of SU(5).

3) The rep of SU(5) on $\Lambda^2 \mathbb{C}^5$ has dim $\frac{5 \cdot 4}{1 \cdot 2} = 10$.

This is good because $1 + 5 + 10 = 16 = \dim F$

where $F$ is the "fermion rep."

Note: $\Lambda^2 (V \oplus W) = \Lambda^2 V \oplus V \otimes W \oplus \Lambda^2 W$

gives

\[
\Lambda^2 \mathbb{C}^5 \cong \Lambda^2 \left( \mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3} \oplus \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1 \right)
\]

\[
\cong \Lambda^2 \left( \mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3} \right) \oplus \left( \mathbb{C}^3 \otimes \mathbb{C} \right) \otimes \left( \mathbb{C} \otimes \mathbb{C}^2 \right) \otimes \left( \mathbb{C}_{-2/3} \otimes \mathbb{C}_1 \right)
\]

\[
\oplus \Lambda^2 \left( \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1 \right)
\]

\[
\cong \Lambda^2 \left( \mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-2/3} \right) \oplus \left[ \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{y/3} \right]
\]

\[
\oplus \Lambda^2 \left( \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_1 \right)
\]

As reps of SU(n), $\Lambda^p \mathbb{C}^n = \Lambda^{n-p} (\mathbb{C}^n^*)$ so

\[
\Lambda^2 \mathbb{C}^3 = \Lambda^1 \mathbb{C}^3^* = \mathbb{C}^3^*
\]

\[
\cong \left[ \mathbb{C}^3^* \otimes \mathbb{C} \otimes \mathbb{C}_{-y/3} \right] \oplus \left[ \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{y/3} \right]
\]

\[
\otimes_{u \mapsto \left( e^u \right) \mapsto \bar{u} \bar{z}}
\]

\[
\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}_z \mapsto \text{anti-} (e^z) = \bar{e}^\bar{z}
\]

\[
\Lambda^2 \mathbb{C}^2 = \Lambda^0 \mathbb{C}^2^* = \mathbb{C}
\]

\[
\uparrow \text{0-forms - like functions}
\]
\[(\Lambda^{p} C^{n})^{\ast} \equiv \Lambda^{n-p} C^{n}\] 

\text{dual of \( p \)-form is \( n-p \)-form} 

Every fermion in \( F \) or its antiparticle shows up, once, in 

\[X = C \oplus C^{5} \oplus \Lambda^{2} C^{5}\]

so fermions and their antiparticles are 

\[F \oplus F^{\ast} = X \oplus X^{\ast}\] 

though, \( F \not\equiv X \)

\[X \oplus X^{\ast} = C \oplus C^{5} \oplus \Lambda^{2} C^{5} \oplus \Lambda^{3} C^{5} \oplus \Lambda^{4} C^{5} \oplus \Lambda^{5} C^{5}\]

\[\Lambda^{0} C^{5} \oplus \Lambda^{1} C^{5} \oplus \Lambda^{2} C^{5} \oplus \Lambda^{3} C^{5} \oplus \Lambda^{4} C^{5} \oplus \Lambda^{5} C^{5}\]

\[= \Lambda C^{5}\]

\text{Moral: Fermions and antifermions form} \( \Lambda C^{5} \) \text{ as a rep of} \ S(\text{U}(3) \times \text{U}(2)). \]