Hypercharge and Weak Isospin

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In the Standard Model, the weak and electromagnetic forces are two aspects of something called the 'electroweak force', which is described by the group $\mathrm{SU}(2) \times \mathrm{U}(1)$. Curiously, it turns out that the familiar concept of 'electric charge' is less fundamental than the concepts of 'weak isospin' and 'hypercharge'. The weak isospin of a particle describes how it transforms under SU(2), while its hypercharge describes how it transforms under U(1). The electric charge is computed in a funny way from these two!

In the following problems, you will examine how this works. The full symmetry group of the Standard Model is

$$\mathrm{ISpin}(3,1) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1),$$

but we will focus on the electroweak force, so we'll ignore $\mathrm{ISpin}(3,1)$ and SU(3) and only think of particles as irreps of $\mathrm{SU}(2) \times \mathrm{U}(1)$. You can read these irreps off the elementary particle chart at the end of this handout.

We begin by introducing bases for $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$:

- The Lie algebra $\mathfrak{su}(2)$ consists of traceless skew-adjoint 2×2 complex matrices, so it has a basis consisting of the matrices $\frac{i}{2}\sigma_j$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- The Lie algebra $\mathfrak{u}(1)$ consists of skew-adjoint 1×1 complex matrices, which are the same as imaginary numbers. For these we choose the basis vector $\frac{i}{3}$.

In quantum theory, observables correspond to self-adjoint operators. Given any unitary representation ρ of SU(2) on some Hilbert space, the above basis of $\mathfrak{su}(2)$ gets mapped by $d\rho$ to skew-adjoint operators on that Hilbert space. Dividing these by i we get self-adjoint operators called the three components of **weak isospin**: I_1, I_2 and I_3. In short:

$$I_j = d\rho(\tfrac{1}{2}\sigma_j)$$

where we have cancelled some factors of i in a slightly underhanded manner.

Similarly, given any unitary representation ρ of U(1) on some Hilbert space, the above basis of U(1) gets mapped by $d\rho$ to a skew-adjoint operator. Dividing this by i we get a self-adjoint operator called **hypercharge**, Y. In short:

$$Y = d\rho(\frac{1}{3}).$$

Digression on History and Terminology

The following stuff is not strictly necessary for doing the homework. But if you want to understand the terms 'isospin' and 'hypercharge' and that weird factor of $\frac{1}{3}$, read this!

The group SU(2) first showed up in physics because it is the double cover of the rotation group SO(3). The matrices $\frac{i}{2}\sigma_j$ serve as a convenient basis for the Lie algebra $\mathfrak{su}(2)$ because they satisfy the commutation relations

$$[\tfrac{i}{2}\sigma_1, \tfrac{i}{2}\sigma_2] = \tfrac{i}{2}\sigma_3 \qquad \text{and cyclic permutations thereof,}$$

which show that $\mathfrak{su}(2)$ is isomorphic to $\mathbb{R}^3$ with its usual cross product:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \text{and cyclic permutations thereof.}$$

While the matrices $\frac{i}{2}\sigma_j$ are skew-adjoint, the matrices $\frac{1}{2}\sigma_j$ are self-adjoint, so they correspond to observables of any quantum system with SU(2) symmetry. When SU(2) is used to describe the rotation symmetries of some system, the observables corresponding to the matrices $\frac{1}{2}\sigma_j$ are called the components of its **angular momentum**. The angular momentum intrinsic to an elementary particle is called its **spin**. The matrices σ_j are called **Pauli matrices** because they were introduced by Pauli in his explanation of the spin of the electron.

Later, Heisenberg proposed SU(2) as a symmetry group for the strong nuclear force. His idea was that this group would explain a symmetry between protons and neutrons: both these particles would really be two states of a single particle called the **nucleon**, which would transform under the spin-$\frac{1}{2}$ representation of SU(2). The proton would be the spin-up state:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2,$$

while the neutron would be the spin-down state:

$$n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2.$$

However, the 'spin' in question here has nothing to do with angular momentum — we're just reusing that word because the same group SU(2) was used to describe rotation symmetries. To avoid confusion, Heisenberg needed another name for this new sort of 'spin'. Since different isotopes of an element differ in how many neutrons they have in their nucleus, he coined the term 'isotopic spin'. Later this got shortened to 'isospin'. So, we call the observables corresponding to $\frac{1}{2}\sigma_j$ in this new context the three components of a particle's **isospin**.

Still later, Glashow, Weinberg and Salam used SU(2) as a symmetry group for the weak nuclear force, and called the observables corresponding to $\frac{1}{2}\sigma_j$ the three components of a particle's **weak isospin**, I_j. There is a close relation between weak isospin and Heisenberg's original isospin: in particular, the isospin of a nucleon is the sum of the weak isospins of the three quarks it is made of. However, weak isospin is now considered to be important for the weak rather than the strong nuclear force.

In short: the same math keeps getting recycled for different physics!

Similarly, U(1) started out being used as a symmetry group for the electromagnetic force. For each integer q there is a unitary irrep ρ of U(1) on $\mathbb{C}$ called the **charge-q irrep**. This is given by

$$\rho(e^{i\theta})\psi = e^{iq\theta}\psi$$

for any unit complex number $e^{i\theta} \in \mathrm{U}(1)$ and any vector $\psi \in \mathbb{C}$. Differentiating with respect to θ and setting $\theta = 0$, we get an irrep $d\rho$ of the Lie algebra $\mathfrak{u}(1)$ on $\mathbb{C}$ with

$$d\rho(i)\psi = iq\psi.$$

The operator $d\rho(i)$ is skew-adjoint, but we can divide it by i to get a self-adjoint operator

$$Q = i^{-1}d\rho(i),$$

or $d\rho(1)$ if we cancel some factors of i in a slightly underhanded way. It's easy to see that

$$Q\psi = q\psi.$$

The observable corresponding to Q is called **electric charge**, and the above equation says that any state of a particle described by the charge-q irrep of U(1) has electric charge equal to q. Using the group U(1) this way gives a nice 'explanation' of the fact that the electric charge is quantized: the charge of any particle is an integer times some smallest charge. However, it doesn't say what this smallest charge actually is!

For a long time people thought that the electron had the smallest possible charge, so they said the electron has charge 1. Actually they said it has charge -1: an unfortunate convention which we can blame on Benjamin Franklin, because he was mixed up about which way the electricity flowed in a current. But what do you expect from someone who *flies a kite with a key hanging on it during a thunderstorm, to attract lightning bolts?* Dumb! But lucky: the next two people to try that experiment were killed.

Much later, people discovered that quarks have electric charges smaller than that of the electron. Measured in units of the electron charge, quark charges are integral multiples of $\frac{1}{3}$. Mathematically it would be nicest to redefine our units of charge so the smallest possible charge is still 1, but people are too conservative to do this, so now the smallest charge is taken to be $\frac{1}{3}$.

Still later, people reused the group U(1) as a symmetry group for the electroweak force, and used the term 'hypercharge' for the observable corresponding to this new U(1). Since hypercharge is closely related to charge, physicists also measure hypercharge in integral multiples of $\frac{1}{3}$.

Here's how we accomodate this foolish factor of $\frac{1}{3}$. For each number y with $3y \in \mathbb{Z}$, there is a unitary irrep ρ of U(1) on $\mathbb{C}$ called the **hypercharge-y irrep**. This is given by

$$\rho(e^{i\theta})\psi = e^{3iy\theta}\psi.$$

Differentiating with respect to θ as before, we get an irrep $d\rho$ of $\mathfrak{u}(1)$ on $\mathbb{C}$ with

$$d\rho(\frac{i}{3})\psi = iy\psi.$$

The operator $d\rho(\frac{i}{3})$ is skew-adjoint, but dividing it by i we get a self-adjoint operator

$$Y = i^{-1}d\rho(\frac{i}{3}),$$

or $d\rho(\frac{1}{3})$ for short. It's easy to see that

$$Y\psi = y\psi.$$

We call the observable corresponding to Y **hypercharge**, and the above equation says that any state of a particle described by the hypercharge-y irrep of U(1) has hypercharge y.

Back to Business

Any particle in the Standard Model corresponds to some unitary irrep of $\mathrm{SU}(2) \times \mathrm{U}(1)$. This is a unitary rep of both SU(2) and of U(1), so we get self-adjoint operators I_1, I_2, I_3 and Y on this irrep, corresponding to weak isospin and hypercharge. The observable **electric charge** is related to these by the mystical formula

$$Q = I_3 + \frac{Y}{2}.$$

Now let's use this to work out the electric charges of all the elementary particles!

I'll do an example: consider the left-handed electron neutrino ν_e^L. As indicated in the chart at the end of this handout, this is the first member of the standard basis of the irrep $\mathbb{C}^2 \otimes \mathbb{C}_{-1}$ of $\mathrm{SU}(2) \times \mathrm{U}(1)$:

$$\nu_e^L = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e^L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that

$$I_3 \nu_e^L = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \nu_e^L.$$

Since the eigenvalue is $\frac{1}{2}$, a physicist reading this equation will say 'the left-handed electron neutrino has $I_3 = \frac{1}{2}$'.

In the chart at the end of this handout, the hypercharge-y irrep of $\mathrm{U}(1)$ is denoted $\mathbb{C}_y$. As explained in the Digression, the hypercharge operator Y acts as multiplication by the number y on any vector in this representation. Since the left-handed electron lives in the hypercharge-(-1) rep, it follows that

$$Y \nu_e^L = -\nu_e^L.$$

Now that we know I_3 and Y for the left-handed electron neutrino, we can use the magic formula to work out its electric charge:

$$Q \nu_e^L = (I_3 + \frac{Y}{2}) \nu_e^L = 0.$$

Since the eigenvalue is 0, the left-handed electron neutrino has electric charge 0. And indeed, this particle is neutral!

Finally, here's where you come in....

1. Use this idea to fill out as much of the following chart as you can. If you know enough representation theory you can do it all! It may help to reread the list of conventions in the previous homework on elementary particles.

type of particle eigenvalue of:	Y	I_3	Q
GAUGE BOSONS			
$g_{rg}, g_{rb}, g_{gr}, g_{gb}, g_{br}, g_{bg}, g_{rr} - g_{bb}, g_{bb} - g_{gg}$			
W_1			
W_2			
W_3			
W_0			
HIGGS BOSON			
H^+			
H^0			
FIRST GENERATION FERMIONS			
Leptons:			
ν_e^L			
e^L			
ν_e^R			
e^R			
Quarks:			
u_r^L, u_g^L, u_b^L			
d_r^L, d_g^L, d_b^L			
u_r^R, u_g^R, u_b^R			
d_r^R, d_g^R, d_b^R			
SECOND GENERATION FERMIONS			
Leptons:			
ν_μ^L			
μ^L			
ν_μ^R			
μ^R			
Quarks:			
c_r^L, c_g^L, c_b^L			
s_r^L, s_g^L, s_b^L			
c_r^R, c_g^R, c_b^R			
s_r^R, s_g^R, s_b^R			
THIRD GENERATION FERMIONS			
Leptons:			
ν_τ^L			
τ^L			
ν_τ^R			
τ^R			
Quarks:			
t_r^L, t_g^L, t_b^L			
b_r^L, b_g^L, b_b^L			
t_r^R, t_g^R, t_b^R			
b_r^R, b_g^R, b_b^R			

2. In problem 1 of the previous homework you may have noticed that for leptons and quarks, the average of the hypercharge of the right-handed ones is equal to the hypercharge of the left-handed one. Use your new-found knowledge to say more about the significance of this fact.

3. What is the sum of the hypercharges of all the fermions in a given generation? To do this right you have to sum over all 16 basis vectors of the fermion rep, e.g. ν_e^L, e^L, ν_e^R, e^R, u_r^L,u_g^L,u_b^L d_r^L,d_g^L,d_b^L, u_r^R,u_g^R,u_b^R, d_r^R,d_g^R,d_b^R.

4. What is the sum of the eigenvalues of I_3 over all the fermions in a given generation?

5. What is the sum of the electric charges of all the fermions in a given generation?

The answers to questions 3-5 are very important in **grand unified theories**. These are theories where $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ is embedded as a Lie subalgebra of some **simple** Lie algebra like $\mathfrak{su}(5)$ or $\mathfrak{so}(10)$: i.e., a Lie algebra with no nontrivial ideals. The fermion rep can only extend to a rep of a simple Lie algebra if the answers to questions 3-5 take a certain special form!

ELEMENTARY PARTICLES IN THE STANDARD MODEL

type of particle	ISpin(3,1) irrep	SU(3) irrep	SU(2) irrep	U
GAUGE BOSONS				
• gluons (SU(3) force carriers):				
$(g_{rg}, g_{rb}, g_{gr}, g_{gb}, g_{br}, g_{bg}, g_{rr} - g_{bb}, g_{bb} - g_{gg})$	massless spin-1	$\mathfrak{su}(3)$	$\mathbb{R}$	$\mathbb{R}$
• SU(2) force carriers:				
(W_1, W_2, W_3)	massless spin-1	$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathbb{R}$
• U(1) force carrier:				
(W_0)	massless spin-1	$\mathbb{R}$	$\mathbb{R}$	$\mathfrak{u}$
HIGGS BOSON				
• Higgs:				
(H^+, H^0)	massless spin-0	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}$
and its antiparticle!				
FIRST GENERATION FERMIONS				
Leptons:				
• left-handed electron neutrino and electron:				
(ν_e^L, e^L)	left-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}$
• right-handed electron neutrino:				
(ν_e^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
• right-handed electron:				
(e^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
and their antiparticles!				
Quarks:				
• left-handed up and down quarks:				
$(u_r^L, u_g^L, u_b^L, d_r^L, d_g^L, d_b^L)$	left-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}$
• right-handed up quark:				
(u_r^R, u_g^R, u_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}$
• right-handed down quark				
(d_r^R, d_g^R, d_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}$
and their antiparticles!				
SECOND GENERATION FERMIONS				
Leptons:				
• left-handed mu neutrino and muon:				
(ν_μ^L, μ^L)	left-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}$
• right-handed mu neutrino:				
(ν_μ^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
• right-handed muon:				
(μ^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
and their antiparticles!				
Quarks:				
• left-handed charm and strange quarks:				
$(c_r^L, c_g^L, c_b^L, s_r^L, s_g^L, s_b^L)$	left-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}$
• right-handed charm quark:				
(c_r^R, c_g^R, c_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}$
• right-handed strange quark				
(s_r^R, s_g^R, s_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}$
and their antiparticles!				
THIRD GENERATION FERMIONS				
Leptons:				
• left-handed tau neutrino and tau:				
(ν_τ^L, τ^L)	left-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}$
• right-handed tau neutrino:				
(ν_τ^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
• right-handed tau:				
(τ^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
and their antiparticles!				
Quarks:				
• left-handed top and bottom quarks:				
$(t_r^L, t_g^L, t_b^L, b_r^L, b_g^L, b_b^L)$	left-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}$
• right-handed top quark:				
(t_r^R, t_g^R, t_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}$
• right-handed bottom quark				
(b_r^R, b_g^R, b_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	