Categorifying the Riemann Zeta Function

John C. Baez, April 13, 2004

In class we discussed the 'skeleton' of a category and sketched the proof that a category is equivalent to any of its skeleta. We'll do this a bit more carefully here, to get some practice in category theory. Then we'll use this fact to work out the cardinalities of some groupoids and the generating functions of some stuff types.

When talking about several categories at once it's good to have names for things that explain which category you're talking about. So, given a category C, let $\mathrm{Ob}(C)$ be the collection of objects in C, and given $x, y \in \mathrm{Ob}(C)$, let $\hom_C(x, y)$ be the collection of morphisms $f: x \to y$ in C.

Definition 1. *A* **subcategory** *of a category D is a category C such that:*

- $\mathrm{Ob}(C) \subseteq \mathrm{Ob}(D)$.
- *For every $x, y \in \mathrm{Ob}(C)$, $\hom_C(x, y) \subseteq \hom_D(x, y)$.*
- *The identity element of $\hom_C(x, x)$ is the same as the identity element of $\hom_D(x, x)$.*
- *The composite of morphisms $f: x \to y$, $g: y \to z$ in C is the same as their composite in D.*

Definition 2. *A subcategory C of a category D is* **full** *if for every $x, y \in \mathrm{Ob}(C)$, $\hom_C(x, y) = \hom_D(x, y)$.*

Definition 3. *A* **skeleton** *of a category D is a full subcategory C of D such that $\mathrm{Ob}(C)$ contains exactly one representative of each isomorphism class of objects in D.*

Whenever C is a subcategory of D, there's an obvious **inclusion** *functor $i: C \to D$ with $i(x) = x$ for every object of C and $i(f) = f$ for every morphism of C.*

1. If C is a subcategory of D, show $i: C \to D$ is faithful.

2. If C is a full subcategory of D, show $i: C \to D$ is full and faithful.

3. If C is a skeleton of D, show $i: C \to D$ is full, faithful, and essentially surjective.

From the homework **Equivalence of Categories***, we can now conclude that a category is equivalent to any of its skeleta!*

4. Suppose C and D are equivalent groupoids. Show that if C is tame then so is D, and that in this case $|C| = |D|$. (Recall that the cardinality $|C|$ of a groupoid C is defined as a sum that may or may not converge; if it converges, we call this groupoid **tame**.)

5. Let Cyc_n be the groupoid whose objects are cyclically ordered n-element sets and whose morphisms are bijections preserving the cyclic ordering. By choosing a skeleton, show that for $n \geq 1$,

$$\mathrm{Cyc}_n \simeq 1/\!\!/\mathbb{Z}_n.$$

Here $\simeq$ means 'equivalent to', and the weak quotient $1/\!\!/G$ is the groupoid having one object and having the group G as morphisms. There are no cyclic orderings on the empty set, so when $n = 0$ Cyc_n is the empty groupoid, i.e. the groupoid with no objects.

6. Use problems 4. and 5. to compute $|\mathrm{Cyc}_n|$.

For any $k \in \mathbb{N}$, let $Z(k)$ be the groupoid of k-tuples of cyclically ordered finite sets, all having the same size. In other words:

- *an object of $Z(k)$ is a k-tuple of cyclically ordered finite sets $(S_1, \ldots, S_k)$, all having the same number of elements;*
- *a morphism $f\colon (S_1, \ldots, S_k) \to (S'_1, \ldots, S'_k)$ is a k-tuple of bijections $f_i\colon S_i \to S'_i$ preserving the cyclic orderings.*

7. Show that

$$Z(k) \simeq \sum_{n \geq 1} 1/\!\!/(\mathbb{Z}_n)^k.$$

8. For which $k \in \mathbb{N}$ is $|Z(k)|$ well-defined, and what does it equal then? Give a closed-form formula for $|Z(2)|$.

9. Let $F\colon Z(k) \to \mathrm{FinSet}_0$ be the functor that forgets everything except the first set in the k-tuple $(S_1, \ldots, S_k)$. This stuff type deserves the name *"being the first of a k-tuple of cyclically ordered subsets, all the same size"*. What is the generating function $|F|(z)$ of this stuff type? What does this equal when $k = 1$ and $z = 1/2$ — and why have you seen this before?

Here's a harder fact: when $k = 2$,

$$|F|(1/2) = \frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2.$$

For extra credit, give a succinct description of the groupoid whose cardinality this is.