$|\mathbb{Z}| = \zeta$

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22 April 2004
QUANTUM GRAVITY SEMINAR
HW#3 (Spring 04)

1) If C is a subcategory of D, then the inclusion functor $i: C \hookrightarrow D$ is faithful.

PROOF: If $f, g \in \hom_C(c,c')$, and $i(f) = i(g)$, then $f = i(f) = i(g) = g$ by definition of the inclusion functor. ■

2) If C is a full subcategory of D, then $i: C \hookrightarrow D$ is full and faithful.

PROOF: A full subcategory is a subcategory, so i is faithful. Moreover, given any $c, c' \in C$ and $f \in \hom_D(i(c), i(c')) = \hom_D(c,c') = \hom_C(c,c')$, f itself is a morphism s.t. $i(f) = f$. So i is full. ■

3) If C is a skeleton of D, then $i: C \hookrightarrow D$ is an equivalence.

PROOF: A skeleton is a full subcategory, so i is full and faithful. Also, if $d \in D$, then by definition of skeleton, $\exists\, \tilde{d} \in C$ s.t. $\tilde{d} \cong d$. That is, $i(\tilde{d}) \cong d$, so i is essentially surjective. A functor is an equivalence iff it's ess. surjective, full, and faithful, so the result follows. ■

4) Let C and D be equivalent groupoids. If C is tame, then D is tame and $|C| = |D|$.

PROOF: The groupoid cardinality is defined (if the sum converges) by

$$|C| = \sum_{[c] \in \underline{C}} \frac{1}{|\mathrm{Aut}(c)|}$$

where the decategorification $\underline{C}$ of a groupoid C is the set of isomorphism classes of C. So we will be done if we establish a one-to-one correspondence between $\underline{C}$ and $\underline{D}$ s.t. if $[c] \in \underline{C}$ corresponds to $[d] \in \underline{D}$ then $|\mathrm{Aut}(c)| = |\mathrm{Aut}(d)|$. Let $F: C \to D$ be an equivalence, with $G: D \to C$ a weak inverse to f. Let $\underline{F}: \underline{C} \to \underline{D}$ be defined by $\underline{F}([c]) = [F(c)]$, and $\underline{G}: \underline{D} \to \underline{C}$ by $\underline{G}([d]) = [G(d)]$. These are well defined, since if $[c] = [c']$

then any isomorphism $f: c \xrightarrow{\sim} c'$ (indeed any morphism, since C is a groupoid) induces an isomorphism $F(f): F(c) \xrightarrow{\sim} F(c')$, so that $[F(c)] = [F(c')]$; similarly for $\underline{G}$. Moreover, $\underline{F}$ & $\underline{G}$ are inverses as set functions, since $\underline{G}(\underline{F}([c])) = [G(F(c))] = [c]$ because $G \circ F$ and 1_C are naturally isomorphic; and similarly $\underline{F}(\underline{G}([d])) = [d]$. So $\underline{C} \underset{\underline{G}}{\overset{\underline{F}}{\rightleftarrows}} \underline{D}$ is a one-to-one correspondence. Finally, we must show that $|Aut(c)| = |Aut(F(c))|$. Since an equivalence is faithful, $f \neq g \in hom(c,c) \implies F(f) \neq F(g) \in hom(F(c), F(c))$. But since C is a groupoid, $hom(c,c) = Aut(c)$, so $|Aut(c)| \leq |Aut(F(c))|$. Since an equivalence is also full, given any $c \in C$ and $f \in Aut(F(c))$, $\exists \tilde{f} \in Aut(c)$ s.t. $F(\tilde{f}) = f$, which implies $|Aut(c)| \geq |Aut(F(c))|$. Hence $|Aut(c)| = |Aut(F(c))|$. We are now ready to do the calculation we're interested in. Suppose C is tame. Then

$$\infty > |C| := \sum_{[c] \in \underline{C}} \frac{1}{|Aut(c)|}$$

$$= \sum_{\underline{F}[c] \in \underline{D}} \frac{1}{|Aut(c)|} \qquad \text{reindex via 1-1 correspondence } [c] \leftrightarrow \underline{F}[c]$$

$$= \sum_{[F(c)] \in \underline{D}} \frac{1}{|Aut(F(c))|} \qquad |Aut(c)| = |Aut(F(c))|$$

$$= \sum_{[d] \in \underline{D}} \frac{1}{|Aut(d)|} \qquad \text{relabel: } d := F(c)$$

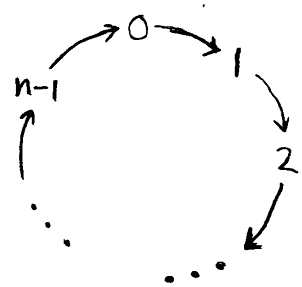
$$=: |D|$$

So $|D|$ is tame. ∎

5) Cyc_n is a groupoid with objects all cyclically ordered n-elt sets, morphisms all cyclic-order-preserving bijections.

<u>Theorem</u>: $\mathrm{Cyc}_n \simeq \frac{1}{\mathbb{Z}/n}$, $\forall n \geq 1$.

<u>Proof</u>: Given any two objects in Cyc_n, there is always a morphism connecting them, so there is only one isomorphism class in Cyc_n. Let C_n be the skeleton of Cyc_n whose only object is the set $n := \{0, 1, 2, \ldots, n-1\}$, cyclically ordered by $(0, 1, \ldots, n-1) \in n!$, i.e.:



To make sure this is a skeleton, we just have to make sure we make it a <u>full</u> subcategory. Thus, the morphisms of C_n will be all bijections $n \to n$ which preserve the cyclic ordering of n. That is, a morphism in C_n is a permutation $\sigma \in n!$ such that

$$(\sigma(0), \sigma(1), \ldots, \sigma(n-1)) = (0, 1, \ldots, n-1) \in n!$$

But then
$$\sigma(0) = j \implies \sigma(1) = j + 1 \bmod n$$
$$\implies \sigma(2) = j + 2 \bmod n$$

and in general $\sigma(i) = j + i \bmod n$, so σ is completely specified by its value at 0, and $\sigma(0) = j \implies \sigma = (0, 1, \ldots, n-1)^j$.

Thus $\mathrm{Aut}_{C_n}(n) = \hom_{C_n}(n, n)$ is isomorphic to the cyclic subgroup of $n!$ generated by $(0, 1, \ldots, n-1)$. That is, $\mathrm{Aut}(n) \cong \mathbb{Z}/n$. So C_n has one object and $\mathbb{Z}/n$ as its group of automorphisms. Since C_n is a skeleton of Cyc_n,

$$\mathrm{Cyc}_n \simeq C_n \simeq \frac{1}{\mathbb{Z}/n} \;. \qquad \blacksquare$$

6.) $\left| \frac{1}{\mathbb{Z}/n} \right| = \frac{1}{|\mathbb{Z}/n|} = \frac{1}{n}$, so $\frac{1}{\mathbb{Z}/n}$ is tame. But $\mathrm{Cyc}_n \simeq \frac{1}{\mathbb{Z}/n}$ by (5) and so Cyc_n is tame and $|\mathrm{Cyc}_n| = \frac{1}{n}$ by (4).

7.) $Z(k)$ is the groupoid with objects all k-tuples of cyclically ordered finite sets of the same cardinality and morphisms k-tuples of bijections preserving cyclic orderings.

Thm: $\quad Z(k) \simeq \sum_{n \geq 1} \frac{1}{(\mathbb{Z}/n)^k}$.

Proof: The groupoid on the right hand side is the disjoint union over $n \in \mathbb{N}^+$ of the groupoids $\frac{1}{(\mathbb{Z}/n)^k}$; $\frac{1}{(\mathbb{Z}/n)^k}$ is a groupoid with one object and $\frac{\mathbb{Z}}{n} \oplus \frac{\mathbb{Z}}{n} \oplus \cdots \oplus \frac{\mathbb{Z}}{n}$ (k summands) as its group of automorphisms. So to complete the proof we just need to see that $Z(k) \simeq \sum_{n \geq 1} (\mathrm{Cyc}_n)^k$ and $(\mathrm{Cyc}_n)^k \simeq \frac{1}{(\mathbb{Z}/n)^k}$. But the latter follows immediately from (5) and the observation that an automorphism of $(n, n, \ldots, n)$ in the skeleton $(C_n)^k$ of $(\mathrm{Cyc}_n)^k$ is just a k-tuple of automorphisms of $n \in C_n$, giving $\frac{\mathbb{Z}}{n} \oplus \cdots \oplus \frac{\mathbb{Z}}{n}$ as the group of automorphisms. The former equivalence is essentially the definition of $Z(k)$, so I'll skip giving a rigorous proof. ∎

8) When the sum converges, we have

$$\begin{aligned} |Z(k)| &= \sum_{n\geq 1} \left| \frac{1}{(\mathbb{Z}/n)^k} \right| \\ &= \sum_{n \geq 1} \frac{1}{|\mathbb{Z}/n|^k} \\ &= \sum_{n\geq 1} \frac{1}{n^k}. \end{aligned}$$

This converges when $k > 1$. That is, $Z(k)$ is tame for $k > 1$, and

$$|Z(k)| = \zeta(k) \qquad k > 1.$$

(Z is a capital ζ, not a capital z!)

Right! So we're proved that decategorification works on Greek as well as Roman letters!!

When $k = 2$, we get:

$$|Z(2)| = \zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

9.)

$$\begin{tikzcd} Z(k) \arrow[d, "F"] \\ \mathrm{FinSet}_0 \end{tikzcd}$$

— forgets everything except the first set in each object

cyclic ordering of S_1

$$\begin{tikzcd} ((S_1, \sigma_1 \in n!), \ldots, (S_k, \sigma_k \in n!)) \arrow[d, mapsto] \\ S_1 \end{tikzcd}$$

The generating function of F is defined by

$$|F|(z) = \sum_{n \in \mathbb{N}} |Z(k)_n|$$

where $Z(k)_n$ is the groupoid of F-stuffed n-elt sets.

Explicitly, $Z(k)_n$ is the groupoid whose objects are all $Z(k)$-objects $(S_1, \dots, S_k)$ such that $F(S_1, \dots, S_k) := S_1$ is an n-elt set, and whose morphisms are all morphisms between these. In fact, by definition of $Z(k)$, if $|S_1| = n$, then $|S_i| = n \ \forall i = 1, 2, \dots, k$. By considerations used in the proof of (7), $Z(k)_n$ is thus a full subcategory of $Z(k)$ equivalent to $\frac{1}{(\mathbb{Z}/n)^k}$. So

$$|F|(z) = \sum_{n \in \mathbb{N}} |Z(k)_n| z^n$$

$$= \sum_{n \in \mathbb{N}} \left| \frac{1}{(\mathbb{Z}/n)^k} \right| z^n$$

$$= \sum_{n \in \mathbb{N}} n^{-k} z^n$$

When $k = 1$ and $z = \frac{1}{2}$, we get

$$|F|(z) = \sum_{n \in \mathbb{N}} \frac{1}{2^n n} = \ln 2$$

Of course we also saw this in the first homework assignment of the quarter (Cyclic & Linear orderings, and Permutations). The reason is just that a 1-tuple of cyclically ordered sets is just a cyclically ordered set, and $\frac{1}{\mathbb{Z}/2} \simeq \frac{1}{2!}$ since $\mathbb{Z}/2 \cong 2!$ as groups.

(Decategorifying $Z = \frac{1}{2!}$ gives $z = \frac{1}{2}$)