Math 260: Categorifying the Riemann Zeta Function

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1. *Subcategories have faithful inclusions.*

Assume that C is a subcategory of D, that is, there is a functor $i: C \to D$ which is an inclusion of objects and morphisms, preserving units and composition. To show that i is faithful, we need to show that i is one-to-one on morphisms, but this is automatically true since i is an inclusion of morphisms.

2. *Full subcategories have full, faithful inclusions.*

If C is a full subcategory of D, then the functor $i: C \to D$ induces a bijection between $\hom_C(c, c')$ and $\hom_D\bigl(i(c), i(c')\bigr)$, so i is clearly onto for morphisms, and hence full in addition to faithful.

3. *Skeletons have essentially surjective, full and faithful inclusions.*

Assume now that, in addition, C is a skeleton of D, and let $d \in D$. There is then exactly one object $c \in C$ in the isomorphism class of d, that is, $i(c) \cong d$ and i is essentially surjective, in addition to full and faithful.

4. *Groupoid cardinality and equivalence.*

If C is a groupoid, then its cardinality is defined by

$$|C| := \sum_{[c]} \frac{1}{|\mathrm{Aut(c)}|}.$$

Suppose that $F: C \to D$ is an equivalence of categories. Then, because F is full and faithful, if $f: c \to c'$ is an isomorphism then $F(f): F(c) \to F(c')$ is, too. Conversely, Let $g: d \to d'$ be an isomorphism in D. Because F is essentially surjective, there are isomorphisms $\alpha_d: F(c) \to d$ and $\alpha_{d'}: F(c') \to d'$, and so there is an isomorphism $\alpha_d g \alpha_{d'}^{-1}: F(c) \to F(c')$. Since F is full, $\alpha_d g \alpha_{d'} = F(f)$ for some $f: c \to c'$, which is an isomorphisms because F is faithful. It follows that

$$|D| = \sum_{[d]} \frac{1}{|\mathrm{Aut(d)}|} = \sum_{[c]} \frac{1}{\bigl|\mathrm{AutF(c)}\bigr|} = \sum_{[c]} \frac{1}{\bigl|\mathrm{Aut(c)}\bigr|} = |C|,$$

and $|D|$ is finite if, and only if, $|C|$ is finite.

5, 6. *Cyclic sets.*

A skeleton of Cyc_n consists of a single cyclically ordered n-element set (since all such are mutually isomorphic) and all bijections of it preserving the cyclic order, which is isomorphic to $\mathbf{Z}_n$. It follows that

$$\mathrm{Cyc}_n \simeq 1//\mathbf{Z}_n.$$

Clearly, then

$$|\mathrm{Cyc}_n| = |1//\mathbf{Z}_n| = \frac{1}{n},$$

since Cyc_n consists of exactly one isomorphism class and the automorphism group $\mathbf{Z}_n$ has order n.

7,8. *The Riemann ζ groupoid.*

The groupoid of k-tuples of cyclically ordered n-element sets is $(\mathrm{Cyc}_n)^k \simeq (1//\mathbf{Z}_n)^k \simeq 1//(\mathbf{Z}_n)^k$. This means that

$$Z(k) \simeq \sum_{n>0} 1//(\mathbf{Z}_n)^k.$$

Then,

$$|Z(k)| = \sum_{n \geq 0} \frac{1}{n^k} = \zeta(k),$$

where ζ is the Riemann zeta function. It follows that $Z(k)$ is tame for all $k > 1$. That $\zeta(2) = \frac{\pi^2}{6}$ is well-known.

9. *The ζ structure type.*

The generating function of $F\colon Z(k) \to E$ is

$$|F|(z) = \sum_{n \geq 0} \frac{z^n}{n^k},$$

which is sometimes denoted $\zeta_k(z)$ in applications to mathematical physics.