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QUANTUM GRAVITY SEMINAR
HW#1 (Spring 04)

LINEAR ORDERINGS, CYCLIC ORDERINGS AND PERMUTATIONS

1) For a linear ordering of n, there are n ways to pick the first element, $n-1$ ways to pick the second, and so on. So there are $n!$ linear orderings of n. Hence

√
$$|L| = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$ as we have observed before.

For $n \neq 0$, the usual way to cyclically order n is to arrange its elements in a circle. This amounts to linearly ordering n and then forgetting which element you started with. This reduces the number of orderings from $n!$ to $\frac{n!}{n} = (n-1)!$, so

√
$$|C| = \sum_{n=1}^{\infty} \frac{(n-1)!\, z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\ln(1-z).$$

using the principal branch of the logarithm.

2) ⓐ $$\frac{d}{dz}|C| = \sum_{n=1}^{\infty} \frac{d}{dz}\frac{z^n}{n} = \sum_{n=1}^{\infty} z^{n-1} = \sum_{n=0}^{\infty} z^n = |L|$$ (or: $\frac{d}{dz}|C| = -\frac{d}{dz}\ln(1-z) = \frac{1}{1-z} = |L|$)

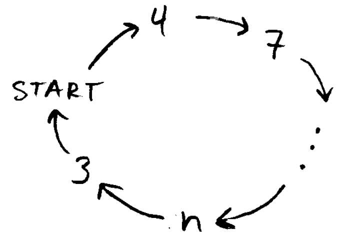
ⓑ $$\frac{d}{dz}|L| = \sum_{n=0}^{\infty} \frac{dz^n}{dz} = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n$$

√
$$\& \quad |L|^2 = \left(\sum_{n=0}^{\infty} z^n\right)\left(\sum_{m=0}^{\infty} z^m\right) = z^0 + (z^0 z^1 + z^1 z^0) + (z^0 z^2 + z^1 z^1 + z^2 z^0) + \cdots$$
$$= z^0 + 2z^1 + 3z^2 + 4z^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (n+1) z^n$$
$$= \frac{d}{dz}|L|$$

(or: $\frac{d}{dz}|L| = \frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2} = |L|^2$)

√ ⓒ $$e^{|C|} = e^{-\ln(1-z)} = \frac{1}{(1-z)} = \sum_{n=0}^{\infty} z^n = |L|.$$

3) A $\frac{DC}{DZ}$-structure on n is a C-structure on $n+1$. That is, to put a $\frac{DC}{DZ}$-structure on the set $\{1, 2, \ldots, n-1\}$, we cyclically order the set $\{1, 2, \ldots, n-1, \text{START}\}$. E.g.



But this obviously defines a linear ordering

$$4 \to 7 \to \cdots \to n \to 3$$

and conversely. In the customary relaxed style of this course, I won't prove naturality in any more rigorous way than to observe that we've made no arbitrary choices. This shows

$$\frac{D}{DZ} C \cong L$$

Similarly, a $\frac{DL}{DZ}$-str. on n is a linear ordering on $n+1$. Thus, to put a $\frac{DL}{DZ}$-str. on $\{1, 2, \ldots, n-1\}$ we linearly order the set consisting of the elts $1, 2, \ldots, n-1$ together with a partition, $\blacksquare$. E.g.:

$$1 \to 2 \to \cdots \to 17 \to 0 \to \blacksquare \to 3 \to \cdots \to 5$$

but this is the same as splitting the set $\{1, 2, \ldots, n-1\}$ in twain and linearly ordering each part:

$$1 \to 2 \to \cdots \to 17 \to 0 \quad \& \quad 3 \to \cdots \to 5$$

Proof by example! $\qquad \frac{D}{DZ} L = L^2.$

Finally, to put an E^c-structure on n, we split n up into disjoint subsets (putting the vacuous structure on the set of subsets), and cyclically ordering the elements of each of the subsets. But this is clearly the same as a permutation of n, decomposed as a product of disjoint cycles, so $E^c \cong P$. However, getting a linear ordering from a permutation requires specifying an element to start with, and there's clearly no canonical way to pick an element. So $P \ncong L$.

Hence $E^c \cong P \ncong L$.

It's actually instructive to give a rigorous proof using the defn of natural isomorphism

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4) Let

$$\frac{1}{2!} =$$  $1+1 \cong 0$ (so $\mathrm{Aut}(\bullet) \cong \mathbb{Z}/2$).

$$\left|C(1/\!\!/2!)\right| = \left|\sum_{n=0}^{\infty} \frac{C_n \times \left(\frac{1}{2!}\right)^n}{n!}\right|$$

$$= \sum_{n=0}^{\infty} \frac{|C_n|\,|1/\!\!/2!|^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(n-1)! \cdot \left(\frac{1}{2}\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n n}$$

$$= -\ln\left(\frac{1}{2}\right)$$

$$= \ln 2$$

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