Linear orders, cyclic orders and permutations

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1, 2. *Generating functions.*

The empty set is vacuously ordered in one way; a 1-element set is trivially ordered in 1 way; and an n-element set can be linearly ordered by choosing a maximal element in one of n ways and linearly ordering the remaining $(n-1)$-element subset. Hence, $|L_n| = n!$ and

$$|L|(z) = \sum_{n\geq 0} \frac{|L_n|}{n!} z^n = \sum_{n\geq 0} z^n = \frac{1}{1-z}.$$

The empty set admits no cyclic orders. If $\{x_1, x_2, \ldots, x_n\}$ is an n-element set, the map

$$x_1 < x_2 < \cdots < x_n \mapsto (x_1\, x_2\, \ldots\, x_n)$$

is an n-to-1 and onto function from linear orders to cyclic permutations Therefore, $|C_n| = \frac{1}{n}|L_n|$ (except for $C_0 = 0$) and

$$|C|(z) = \sum_{n\geq 1} \frac{z^n}{n} = \ln\left(\frac{1}{1-z}\right).$$

It follows that

$$\frac{d}{dz}|C|(z) = |L|(z); \qquad \frac{d}{dz}|L|(z) = |L|^2(z); \qquad \text{and} \quad e^{|C|(z)} = |L|(z).$$

3. *Structure type isomorphisms.*

A $\frac{D}{DZ}C$-structure on an n-element set is a pointed C-structure on an $(n+1)$-element set. But this is equivalent to a linear order starting with the basepoint and increasing by the action of the cyclic permutation, so it provides a linear order of the original n-element set. Conversely, a linear order on an n-element set can be turned into a pointed cyclic order on an $(n+1)$-element set by inserting the basepoint between the maximal and minimal elements for the linear order. Hence,

$$\frac{D}{DZ}C \cong L.$$

Similarly, a pointed linear order on an $(n+1)$-element set is equivalent to an ordered partition of an n-element set into two linearly ordered (possibly empty) subsets. Conversely, given an ordered pair of linealy ordered sets having jointly n elements, one can insert a basepoint between the first and second sets and obtain a pointed linear order on an $(n+1)$-element set. Therefore,

$$\frac{D}{DZ}L \cong L^2$$

Finally, observe that a cyclic permutation is a "connected permutation" and that exponentiation of structure types corresponds to building possibly disconnected structures from connected ones. Indeed, any permutation has a unique cycle decomposition, which is a partition of the original set into a collection of orbits of cyclic permutations. Hence,

$$E^C \cong P.$$

Although $|P| = |L|$, a linear order is not a permutation. For each finite set, the permutations on it form a group P_n, and the linear orders form a P_n-torsor. There can be no canonical isomorphism between the two.

4. *The groupoid cardinality of $C(1//2!)$.*

We have

$$|C(1//2!)| = \ln\left(\frac{1}{1-\frac{1}{2}}\right) = \ln 2.$$