Linear Orderings, Cyclic Orderings and Permutations

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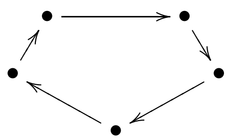
A **linear ordering** *of a set S, also called a* **total ordering**, *is a binary relation $<$ on S that is:*

- *irreflexive ($x \not< x$),*
- *asymmetric ($x < y \Longrightarrow y \not< x$),*
- *transitive ($x < y$ & $y < z \Longrightarrow x < z$)*
- *and linear ($x \neq y \Longrightarrow x < y$ or $y < x$).*

In pictures, a linear ordering looks something like this:

A 'cyclic ordering', on the other hand, looks like this:



More formally, we can define a **cyclic ordering** *of the finite set S to be a permutation $\sigma: S \to S$ with exactly one orbit. The permutation maps each element of S to the 'next one on the cycle'. We can also define a cyclic ordering to be an equivalence class of linear orderings, where the linear ordering of $\{x_1, \ldots, x_n\}$ with*

$$x_1 < x_2 < \cdots < x_{n-1} < x_n$$

is equivalent to the total ordering with

$$x_n < x_1 < x_2 < \cdots < x_{n-1}.$$

("And the last shall be first." — Matthew 19.) However, this definition is valid only if S is nonempty; the empty set has no cyclic ordering (because its unique permutation has zero orbits).

Let L be the structure type "being a linearly ordered finite set", and let C be the structure type "being a cyclically ordered finite set". There are some nice relations between these two structure types.

1. Compute the generating functions $|L|$ and $|C|$ directly, by counting the number of linear orderings and cyclic orderings on an n-element set.

To linearly order a set with n elements, first pick the first element (in n possible ways), then pick the next element (in $n-1$ possible ways), then ... then pick the last element (in 1 possible way). Thus the number of ways to linearly order a set with n elements is $n!$. Therefore,

$$|L| = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

To cyclically order a set with $n > 0$ elements, linearly order it in $n!$ ways if you like; but then notice that this is overdetermined by a factor of n, since we should not have been able to tell which of n positions is the starting position. Thus the number of ways to cyclically order a set with $n > 0$ elements is $(n-1)!$. On the other hand, the empty set can't be cyclically ordered, so the number of ways to cyclically order a set with 0 elements is 0. Therefore,

$$|C| = \sum_{n=1}^{\infty} \frac{(n-1)!}{n!} z^n = \sum_{n=1}^{\infty} \frac{1}{n} z^n.$$

I could find a closed form for $|C|$ now, but it's easier to do this in the context of question 2..

2. Using 1. show that

$$\frac{d}{dz}|C| = |L|$$
$$\frac{d}{dz}|L| = |L|^2$$
$$e^{|C|} = |L|.$$

Differentiating term by term,

$$\frac{d}{dz}|C| = \sum_{n=1}^{\infty} \frac{d}{dz} \frac{1}{n} z^n = \sum_{n=1}^{\infty} z^{n-1} = \sum_{n=0}^{\infty} z^n = |L|.$$

Now using a closed form,

$$\frac{d}{dz}|L| = \frac{d}{dz} \frac{1}{1-z} = \frac{1}{(1-z)^2} = |L|^2.$$

Now combining these facts,

$$\frac{d}{dz}|L| = |L|^2 = |L| \frac{d}{dz}|C|,$$

or

$$\frac{d}{dz}|C| = \frac{1}{|L|} \frac{d}{dz}|L| = \frac{d}{dz} \ln |L|.$$

Also $|C|(0) = |C|_0 = 0$ and $|L|(0) = |L|_0 = 1$, so $|C|(0) = \ln |L|(0)$. Therefore, $|C| = \ln |L|$, or $|L| = e^{|C|}$. (Note that $|L|$ has an inverse and a logarithm, since $|L|_0 > 0$.)

I can now write down a closed form for $|C|$;

$$|C| = \ln |L| = \ln \frac{1}{1-z} = -\ln(1-z).$$

3. Do the above equations between generating functions come from natural isomorphisms between the structure types? Show that

$$\frac{D}{DZ} C \cong L$$

and

$$\frac{D}{DZ} L \cong L^2$$

but

$$E^C \not\cong L.$$

Hint: Hint: for the last one, let P be the structure type "being a finite set equipped with a permutation of its elements". Show that $E^C \cong P$ and $P \ncong L$.

To place a $\frac{D}{DZ}C$ structure on a set S, adjoin a point and then cyclically order the result $S+$. By starting with the adjoined point and ordering from there, this defines a linear order on S. Conversely, if S has been linearly ordered, then we may cyclically order $S+$ by mapping each point to the next in order, with the extra point serving on the boundary. These transformations are inverses of each other, so $\frac{D}{DZ}C \cong L$.

To place a $\frac{D}{DZ}L$ structure on a set S, adjoin a point and then linearly order the result $S+$. The new point will break S up into pieces (one before the point and one after it), each of which will be linearly ordered, giving an L^2 structure on S. Conversely, if S has an L^2 structure, then linearly order $S+$ by going through the one subset of L in order, then the new point, then the other subset of L in order. These transformations are inverses of each other, so $\frac{D}{DZ}L \cong L^2$.

At this point, I should be able to calculate $\frac{D}{DZ}C = \frac{D}{DZ}\operatorname{LN} L$ and $C(0) \cong \operatorname{LN} L(0)$, although LN hasn't yet been covered fully in class. Even so, this is insufficient to conclude that $C \cong \operatorname{LN} L$ and then $E^C \cong L$, since differential equations don't have unique solutions. Indeed, $E^C \ncong L$, as stated.

For, to place an E^C structure on a set S, partition S and cyclically order each piece. Since each point x of S belongs to a unique piece, whose cyclic ordering assigns x a value, this defines a single permutation on all of S. Conversely, given a permutation p on S, the orbits of p form a partition of S, each piece of which has a single orbit of the partition, which is a cyclic order. These transformations are inverses of each other, so $E^C \cong P$.

Now suppose that $P \cong L$ through a natural isomorphism H. Then in particular, $P_2 \cong L_2$ through a bijection H_2. Now, P_2 is the set of permuations of $2 = \{0, 1\}$; these permutations are $i := \{(0, 0), (1, 1)\}$ and $t := \{(0, 1), (1, 0)\}$. On the other hand, L_2 is the set of linear orderings of $2 = \{0, 1\}$; these linear orderings are $f := \{(0, 1)\}$ and $b := \{(1, 0)\}$. Now, H_2 maps i to either f or b, not both. Either way, it must respect the automorphisms of 2, since it comes from a natural isomorphism. In particular, H_2 must respect the involution τ that swaps 0 to 1. (Yes, t and τ are literally the same thing, but they are playing very different roles here.) Now, P_τ fixes i, but L_τ swaps f with b. This is a contradiction; therefore, $P \ncong L$.

P and L are an interesting pair of structure types. Even though a permutation is very different from a linear order:

$$P \ncong L$$

there are just as many permutations of a finite set as linear orders on it:

$$|P| = |L|$$

and we've seen above that both can be defined in terms of cyclic orderings:

$$P \cong E^C, \qquad L \cong \frac{DC}{DZ}.$$

4. Let $1//2!$ be the groupoid with one object and $\mathbb{Z}_2$ as the group of automorphisms of this object, so that

$$|1//2!\,| = 1/2.$$

Calculate the groupoid cardinality of $C(1//2!)$. This is the groupoid of 'half-colored cyclically ordered finite sets'.

Knowing the generating function for C, this is easy:

$$|C(1//2!)| = |C|(|1//2!|) = -\ln(1 - 1/2) = \ln 2.$$