**Theorem:** A functor $F: C \longrightarrow D$ is an equivalence if and only if it is essentially surjective, full and faithful.

**Proof:** $\boxed{\Longrightarrow}$ Assume first that $F$ is an equivalence. Then we have a functor $G: D \longrightarrow C$ and natural isomorphisms $\alpha: FG \overset{\sim}{\Longrightarrow} 1_C$, $\beta: GF \overset{\sim}{\Longrightarrow} 1_D$:

$$\begin{tikzcd}
& D \arrow[rr, "1_D"] \arrow[dr, "G"] \arrow[d, phantom, "\alpha \Downarrow \wr"] & \arrow[d, phantom, "\wr \Uparrow \beta"] & D \\
C \arrow[ur, "F"] \arrow[rr, "1_C"'] & {} & C \arrow[ur, "F"'] &
\end{tikzcd}$$

To show that $F$ is essentially surjective, pick an object $d \in D$. We must show that $d$ lifts to $\tilde{d} \in C$, s.t. $F(\tilde{d}) \cong d$. The obvious choice is to let $\tilde{d} := G(d)$. Then

$$F(\tilde{d}) = F(G(d)) = (GF)(d),$$

but then since $\alpha$ is a natural isomorphism, we get an isomorphism

$$\begin{tikzcd}
\alpha_{\tilde{d}}: F(\tilde{d}) \arrow[r, "\sim"] \arrow[d, equal] & d \arrow[d, equal] \\
(GF)(d) & 1_D(d)
\end{tikzcd}$$

In particular, $F(\tilde{d}) \cong d$.

Next we show $F$ is full. Let $c, c'$ be $C$-objects. Given any morphism $f: F(c) \longrightarrow F(c')$ in $D$, we must show there's a morphism $\tilde{f}: c \longrightarrow c'$ in $C$ s.t. $F(\tilde{f}) = f$. First consider the morphism $G(f): (FG)(c) \longrightarrow (FG)(c')$. The problem with this as a candidate for $\tilde{f}$ is that $(FG)(c)$ may not equal $c$; however, they are isomorphic:

$$\begin{tikzcd}
(FG)(c) \arrow[r, "G(f)"] \arrow[d, "\alpha_c"', "\wr"] & (FG)(c') \arrow[d, "\alpha_{c'}"', "\wr"] \\
c & c'
\end{tikzcd}$$

So to get a morphism from $c$ to $c'$, we just fill in the missing side in the obvious way, letting $\tilde{f} = \alpha_c^{-1} G(f) \alpha_{c'}$ :

$$\begin{tikzcd}
(FG)(c) \arrow[r, "G(f)"] \arrow[d, "\alpha_c"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & (FG)(c') \arrow[d, "\wr"', "\alpha_{c'}"] \\
c \arrow[r, "\tilde{f}"'] & c'
\end{tikzcd}$$

Definition of $\tilde{f}$

Note that this looks very much like the naturality square of $\alpha$ on $\tilde{f}$ :

$$\begin{tikzcd}
(FG)(c) \arrow[r, "FG(\tilde{f})"] \arrow[d, "\alpha_c"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & (FG)(c') \arrow[d, "\wr"', "\alpha_{c'}"] \\
c \arrow[r, "\tilde{f}"'] & c'
\end{tikzcd}$$

Since both of these diagrams commute and the vertical arrows are invertible, we can paste them together along the edge $\tilde{f}: c \longrightarrow c'$ and get

$$\begin{tikzcd}
(FG)(c) \arrow[r, "G(f)"] \arrow[d, "\alpha_c"'] & (FG)(c') \arrow[d, "\alpha_{c'}"] \\
c \arrow[d, "\alpha_c^{-1}"'] \arrow[r, phantom, "\circlearrowleft"] & c' \arrow[d, "\alpha_{c'}^{-1}"] \\
(FG)(c) \arrow[r, "FG(\tilde{f})"'] & (FG)(c')
\end{tikzcd}$$

which just says:

$$G(f) = (FG)(\tilde{f}). \qquad (*)$$

We can use this result to show that $F(\tilde{f}) = f$ by considering the naturality of $\beta : GF \Longrightarrow 1_D$ on both $f$ and $F(\tilde{f})$ :

$$\begin{tikzcd}
(GF)(F(c)) \arrow[r, "(GF)(f)"] \arrow[d, "\beta_{F(c)}"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & (GF)(F(c')) \arrow[d, "\beta_{F(c')}"] \\
F(c) \arrow[r, "f"'] & F(c')
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
(GF)(F(c)) \arrow[r, "(GF)(F(\tilde{f}))"] \arrow[d, "\beta_{F(c)}"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & (GF)(F(c')) \arrow[d, "\wr"', "\beta_{F(c')}"] \\
F(c) \arrow[r, "F(\tilde{f})"'] & F(c')
\end{tikzcd}$$

Note that our previous result (*) implies that the morphisms on the tops of these diagrams are actually the same:

$$(GF)(f) = F(G(f)) \underset{(*)}{=} F((FG)(\tilde{f})) = F(G(F(\tilde{f}))) = (GF)(F(\tilde{f})).$$

So we can paste these together to get:

$$\begin{tikzcd}
F(c) \arrow[rr, dashed, "f"] \arrow[drr, phantom, "\circlearrowleft"] & & F(c') \\
(GF)(F(c)) \arrow[u, "\beta_{F(c)}", "\wr"'] \arrow[d, "\beta_{F(c)}"', "\wr"] \arrow[rr, "(GF)(f)", "=(GF)(F(\tilde{f}))"'] \arrow[drr, phantom, "\circlearrowleft"] & & (GF)(F(c')) \arrow[u, "\wr", "\beta_{F(c')}"'] \arrow[d, "\wr"', "\beta_{F(c')}"] \\
F(c) \arrow[rr, "F(\tilde{f})"'] & & F(c')
\end{tikzcd}$$

Cancelling the $\beta_{F(c)}$'s and $\beta_{F(c')}$'s, we get $f = F(\tilde{f})$ as desired, so $F$ is full.

Showing that $F$ is faithful is simpler. Again let $c, c' \in C$. Suppose $f, g \in \hom(c,c')$ are such that $F(f) = F(g) \in \hom(F(c), F(c'))$. Then applying $G: D \to C$ to the diagram

$$\begin{tikzcd}
F(c) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\|"] & F(c')
\end{tikzcd}$$

and using naturality of $\alpha$, we obtain a cylinder:

$$\begin{tikzcd}
(FG)(c) \arrow[rr, bend left, "FG(f)"] \arrow[rr, bend right, "FG(g)"'] \arrow[rr, phantom, "\|"] \arrow[d, "\alpha_c"', "\wr"] \arrow[drr, phantom, "\circlearrowleft"] & & (FG)(c') \arrow[d, "\wr"', "\alpha_{c'}"] \\
c \arrow[rr, bend left, dashed, "f"] \arrow[rr, bend right, "g"'] & & c'
\end{tikzcd}$$

Since the front and back commute and $\alpha_c$ is invertible, $f = g$ by the obvious diagram chase, and hence $F$ is faithful.

$\boxed{\Leftarrow}$ To prove the converse, suppose we are given $F: C \to D$ which is essentially surjective, full and faithful. We must construct a functor $G: D \to C$ and natural isomorphisms $\alpha: FG \xRightarrow{\sim} 1_C$, $\beta: GF \Rightarrow 1_D$.

First, given $d \in D$, let $G(d) = \tilde{d}$ be the weak lifting of $d$ to $C$ via essential surjectivity, so that $F(\tilde{d}) \cong d$. Call the isomorphism which carries $F(\tilde{d})$ to $d$ by the name $\beta_d$. We have specified what $G$ should do to objects; we should now say what it does to morphisms. Let $f \in \hom(d, d')$ be a morphism in $D$. Lift $d$ & $d'$ weakly to $\tilde{d} =: G(d)$ and $\tilde{d}' =: G(d')$. Then we have the following diagram:

$$\begin{tikzcd}
F(\tilde{d}) \arrow[d, equal] & F(\tilde{d}') \arrow[d, equal] \\
F(G(d)) \arrow[d, "\beta_d"', "\wr"] & F(G(d')) \arrow[d, "\beta_{d'}", "\wr"'] \\
d \arrow[r, dashed, "f"] & d'
\end{tikzcd}$$

Then the natural way — for indeed, this definition will guarantee naturality of $\beta$ — to define $G(f)$ is as the lifting of this composite via the fact that $F$ is **full**:

$$G(f) := \widetilde{\beta_d f \beta_{d'}^{-1}}.$$

That is, since $\tilde{d}, \tilde{d}' \in C$, we can let $G(f): \tilde{d} \to \tilde{d}'$ be a morphism such that $F(G(f)) = \beta_d f \beta_{d'}^{-1}$. Note that since $F$ is also faithful, such $G(f)$ is also unique. Moreover, by construction $F(G(f))$ fills in the missing side of the above square:

$$\begin{tikzcd}
F(G(d)) \arrow[r, dashed, "F(G(f))"] \arrow[d, "\beta_d"', "\wr"] & F(G(d')) \arrow[d, "\beta_{d'}", "\wr"'] \\
d \arrow[r, "f"] & d'
\end{tikzcd}$$

This means that $\beta$ so defined will be a natural transformation of functors provided that $G$ is in fact a functor. This is indeed the case:

To see that $G$ is a functor, we first show that for any object $d \in D$, $G(1_d) = 1_{G(d)}$. Applying $\beta$ to $1_d$ we get

$$\begin{tikzcd} F(G(d)) \arrow[r, "F(G(1_D))"] \arrow[d, "\beta_d"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & F(G(d)) \arrow[d, "\wr"', "\beta_d"] \\ d \arrow[r, "1_d"] & d \end{tikzcd}$$

or:

$$\begin{tikzcd} F(G(d)) \arrow[r, "F(G(1_D))"] \arrow[r, bend right=70, "1_{F(G(d))}"'] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & F(G(d)) \end{tikzcd}$$

(by cancelling $\beta_d 1_d \beta_d^{-1}$)

But since $F$ is a functor, $1_{F(G(1_D))} = F(1_{G(d)})$ so we have $F(G(1_D)) = F(1_{G(d)})$. Since $F$ is faithful, we conclude that $G(1_D) = 1_{G(d)}$. We still have to show that $G$ preserves composition. Let $f \in \hom(d, d')$ and $f' \in \hom(d', d'')$. Then the diagram

$\beta$ applied to $f: d \to d'$

$\beta$ applied to $f': d' \to d''$

$$\begin{tikzcd} F(G(d)) \arrow[r, "F(G(f))"] \arrow[d, "\beta_d"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & F(G(d')) \arrow[r, "F(G(f'))"] \arrow[d, "\beta_{d'}"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & F(G(d'')) \arrow[d, "\wr"', "\beta_{d''}"] \\ d \arrow[r, "f"] \arrow[rr, bend right=30, "ff'"'] & d' \arrow[r, "f'"] \arrow[d, phantom, "\parallel"] & d'' \\ & \circlearrowleft & \\ F(G(d)) \arrow[uuu, "\beta_d", "\wr"'] \arrow[rr, "F(G(ff'))"] & & F(G(d)) \arrow[uuu, "\beta_{d''}"'] \end{tikzcd}$$

$\beta$ applied to $ff': d' \to d''$

gives us $F(G(f))\,F(G(f')) = F(G(ff'))$, by collapsing the vertical sides. Since $F$ is a functor, this means $F(G(f)\,G(f')) = F(G(ff'))$; but since $F$ is faithful we then get $G(f)\,G(f') = G(ff')$. So $G$ preserves both identities and composition, hence is a functor.

So far, we have a functor $G: D \longrightarrow C$ and a natural isomorphism $\beta: GF \overset{\sim}{\Rightarrow} 1_D$ ($\beta$ being defined by the assignment of $\beta_d$ to each $d \in D$ — this is natural by construction and an isomorphism since each $\beta_d$ is). We have yet to define $\alpha: FG \Longrightarrow 1_e$. Let's do this now. Given any $c \in C$, we want an isomorphism $\alpha_c: G(F(c)) \xrightarrow{\sim} c$. What we already have is an isomorphism $\beta_{F(c)}: F(G(F(c))) \xrightarrow{\sim} F(c)$. Since $F$ is full, this lifts (weakly) to an isomorphism

$$\alpha_c := \widetilde{\beta_{F(c)}} : G(F(c)) \xrightarrow{\sim} c.$$

Moreover, this $\alpha_c$ is uniquely given by the lifting condition $F(\alpha_c) = \beta_{F(c)}$, since $F$ is faithful. To see that the assignment $c \mapsto \alpha_c$ defines a natural isomorphism, pick $f \in \hom(c, c')$ and draw the square we hope commutes:

$$\begin{tikzcd}
G(F(c)) \arrow[r, "G(F(f))"] \arrow[d, "\alpha_c"', "\wr"] \arrow[dr, phantom, "\circlearrowleft ?"] & G(F(c')) \arrow[d, "\wr"', "\alpha_{c'}"] \\
c \arrow[r, "f"] & c'
\end{tikzcd}$$

Apply $F$:

$$\begin{tikzcd}
F(G(F(c))) \arrow[r, "F(G(F(f)))"] \arrow[d, "F(\alpha_c) = \beta_{F(c)}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(G(F(c'))) \arrow[d, "\beta_{F(c')} = F(\alpha_{c'})"] \\
F(c) \arrow[r, "F(f)"'] & F(c')
\end{tikzcd},$$

and we get a commuting square by naturality of $\beta$. But commutativity of the second square implies commutativity of the first, since $F$ is faithful.

Whew!! ∎