Math 260: Equivalence of Categories

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Suppose $F\colon C \to D$ is an equivalence of categories. This means there is a functor $G\colon D \to C$ which is a two-sided inverse of F up to natural isomorphism, that is, there are natural isomorphisms $\alpha\colon GF \Rightarrow 1_C$ and $\beta\colon FG \Rightarrow 1_D$. (Note: here we are using the convention in which the result of first applying F and then G is denoted GF.) This means that, for each $f\colon c \to c'$ in C and for each $g\colon d \to d'$ in D, the following diagrams commute:

$$\begin{array}{ccc} GF(c) & \xrightarrow{\alpha_c} & c \\ \downarrow{\scriptstyle GF(f)} & & \downarrow{\scriptstyle f} \\ GF(c') & \xrightarrow{\alpha_{c'}} & c' \end{array} \qquad \begin{array}{ccc} FG(d) & \xrightarrow{\beta_d} & d \\ \downarrow{\scriptstyle FG(g)} & & \downarrow{\scriptstyle g} \\ FG(d') & \xrightarrow{\beta_{d'}} & d' \end{array}$$

We need to show that F is faithful, full and essentially surjective.

Suppose that $f, f'\colon c \to c'$ are such that $F(f) = F(f')$. Then $GF(f) = GF(f')$ and, by naturality of α,

$$f = \alpha_{c'} GF(f) \alpha_c^{-1} = \alpha_{c'} GF(f') \alpha_c^{-1} = f'.$$

This shows that F is faithful. Similarly, G is faithful.

Now let $c, c' \in C$ and let $g\colon F(c) \to F(c')$ in D. Then, $G(g)\colon GF(c) \to GF(c')$. By naturality of α, if $f = \alpha_{c'} G(g) \alpha_c^{-1}\colon c \to c'$, we have $G(g) = GF(f)$. Then, $FG(g) = FGF(f)$ and, since F and G are faithful, $g = F(f)$ and F is full.

Finally, suppose $d \in D$. The isomorphism $\beta_d\colon FG(d) \to d$ makes F essentially surjective, with $G(d)$ in the essential preimage of d.

For the converse, suppose that $F\colon C \to D$ is essentially surjective, full and faithful. This means that: for each $d \in D$, there is a $c = G(d) \in C$ and an isomorphism $\beta_d\colon F(c) \to d$; and, for each $c, c' \in C$, the map $F\colon C(c, c') \to D\big(F(c), F(c')\big)$ is a bijection. We need to construct a two-sided weak inverse of $F\colon C \Rightarrow D$.

We already have defined G on objects. Now, given $g\colon d \to d'$ in D, we want to define $G(g)\colon G(d) \to G(d')$. We have a bijection between hom-sets $F\colon \hom_C\big(G(d), G(d'))\big) \to \hom_D\big(FG(d), FG(d')\big)$. Observe that $g \mapsto \beta_{d'} g \beta_d^{-1}$ is a bijection from $\hom_D(d, d')$ to $\hom_D\big(FG(d), FG(d')\big)$, and define a map of morphisms $G(g) = F^{-1}(\beta_{d'} g \beta_d^{-1})$. Given a $g'\colon d' \to d''$, we have

$$G(g'g) = F^{-1}(\beta_{d''} g' g \beta_d^{-1}) = F^{-1}(\beta_{d''} g' \beta_{d'}^{-1} \beta_{d'} g \beta_d^{-1}) = F^{-1}(\beta_{d''} g' \beta_{d'}^{-1}) F^{-1}(\beta_{d'} g \beta_d^{-1}) = G(g')G(g),$$

so that $G\colon D \to C$ so defined is a functor.

The definition $G(g) = F^{-1}(\beta_{d'} g \beta_d^{-1})$ implies that

$$\begin{array}{ccc} FG(d) & \xrightarrow{\beta_d} & d \\ \downarrow{\scriptstyle FG(g)} & & \downarrow{\scriptstyle g} \\ FG(d') & \xrightarrow{\beta_{d'}} & d' \end{array}$$

commutes, defining a natural isomorphism $\beta\colon FG \Rightarrow 1_D$.

Finally, for every $c \in C$ we have an isomorphism $\beta_{F(c)}\colon FGF(c) \to F(c)$. Since F is full and faithful, we can define $\alpha_c = F^{-1}(\beta_{F(c)})\colon GF(c) \to c$, which is also an isomorphism. Now, applying the definition of $G(g) = F^{-1}(\beta_{d'} g \beta_d^{-1})$ to $g = F(f)\colon d = F(c) \to F(c') = d'$, we get

$$GF(f) = F^{-1}(\beta_{F(c')} F(f) \beta_{F(c)}^{-1}) = \alpha_{c'} f \alpha_c^{-1},$$

so

$$\begin{array}{ccc} GF(c) & \xrightarrow{\alpha_c} & c \\ \downarrow{\scriptstyle GF(f)} & & \downarrow{\scriptstyle f} \\ GF(c') & \xrightarrow{\alpha_{c'}} & c' \end{array}$$

commutes, and $\alpha\colon GF \Rightarrow 1_C$ is a natural isomorphism.