Toby Bartels* **Equivalence of Categories** **2004 April 15**

Notation

John has confused everybody by writing functor *application* in the traditional Leibniz fashion while simultaneously writing functor *composition* in the intuitive antiLeibniz fashion. Thus I feel justified in introducing yet another notation. It will not conflict with anything above; it is essentially antiLeibniz but does not use parentheses, so there will be no confusion with John's Leibniz notation using parentheses. (This is easy to do, of course, by treating application as a special case of composition; but that is not pedagogically sound. My notation continues to distinguish these.)

Specifically, if F is an operator acting on x, then the result of the application is denoted x^F; I deliberately mean to evoke the algebraic intuition surrounding exponentiation. For example, if G acts on x^F, then the result must be denoted $(x^F)^G$; but the notation suggests a simplification to x^{FG}. Thus FG is the composition of F and G, doing F first and then doing G. Note that I will use this superscript notation regardless of whether the argument x is an object or a morphism and regardless of whether the operator F is a functor or a natural transformation. Thus the subscript notation for natural transformation application is also abolished, restoring the proper symmetry between functors and natural transformations. Finally, I shall find it convenient to abbreviation $(x^\alpha)^{-1}$ as $x^{-\alpha}$ when x^α is an isomorphism. (Thus if α is a natural isomorphism, then $-\alpha$ is the inverse natural isomorphism; although I'll have no call to refer to $-\alpha$ by itself.)

Note that the notation *internal* to a category is unchanged; morphism composition is written antiLeibniz as John did consistently, and that's all that there is to that; there is no notion of morphism application. That said, the notation is still suggestive of true facts; for example, the requirement that a functor preserve composition is the equation $(fg)^F = f^F g^F$.

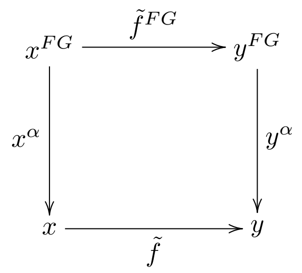
Essential surjectivity

In this section, I prove that equivalences are essentially surjective. So let C and D be categories, let the functor $F: C \to D$ be an equivalence (with weak inverse $G: D \to C$ and natural isomorphisms $\alpha: FG \Rightarrow 1_C$ and $\beta: GF \Rightarrow 1_D$), and let x be an object of D. Then x^G is an object of C; let $\tilde{x}$ be this. Then x^β is an isomorphism in D from $x^{GF} = \tilde{x}^F$ to $x^{1_D} = x$. Therefore, $\tilde{x}^F \cong x$ as desired.

Fullness

In this section, I prove that equivalences are full. So let C, D, and F (with G, α, and β) be as above, let x and y be objects in C, and let $f: x^F \to y^F$ be a morphism in D. Then $f^G: x^{FG} \to y^{FG}$ is a morphism in C; let $\tilde{f}$ be the morphism $x^{-\alpha} f^G y^\alpha := (x^\alpha)^{-1} f^G y^\alpha : x \to y$.

Now consider the naturality diagram for α as applied to $\tilde{f}$:



This diagram is basically the definition of $\tilde{f}$, only with $\tilde{f}^{FG}$ in the place of f^G. But because x^α and y^α are isomorphisms, it thus follows that $\tilde{f}^{FG}$ and f^G are in fact equal. Using equations only, I can also calculate as follows, starting with the naturality condition:

$$\begin{aligned}
\tilde{f}^{FG} y^{\alpha} &= x^{\alpha} \tilde{f} \\
&= x^{\alpha} x^{-\alpha} f^{G} y^{\alpha} \\
&= f^{G} y^{\alpha}; \\
\tilde{f}^{FG} &= f^{G}.
\end{aligned}$$

Now, I *want* to get $\tilde{f}^{F} = f$, but I can't simply drop G. Instead, let me apply the weak inverse of G, which is F again, to get $\tilde{f}^{FGF} = f^{GF}$. This suggests that I should look at the naturality diagram for β as applied to f:

$$\begin{array}{ccc}
x^{FGF} & \xrightarrow{\;f^{GF}\;} & y^{FGF} \\
{\scriptstyle x^{F\beta}}\big\downarrow & & \big\downarrow{\scriptstyle y^{F\beta}} \\
x^{F} & \xrightarrow[\;f\;]{} & y^{F}
\end{array}$$

Compare this to the naturality diagram for β as applied to $\tilde{f}^{F}$:

$$\begin{array}{ccc}
x^{FGF} & \xrightarrow{\;\tilde{f}^{FGF}\;} & y^{FGF} \\
{\scriptstyle x^{F\beta}}\big\downarrow & & \big\downarrow{\scriptstyle y^{F\beta}} \\
x^{F} & \xrightarrow[\;\tilde{f}^{F}\;]{} & y^{F}
\end{array}$$

These diagrams are identical except for the horizontal arrows. (The reason for this is that f and $\tilde{f}^{F}$ have the same domain and codomain.) But I've already calculated that the top arrows are equal. Since the vertical arrows are invertible, it follows that the bottom arrows are also equal. That is, $f = \tilde{f}^{F}$, so the functor F is full.

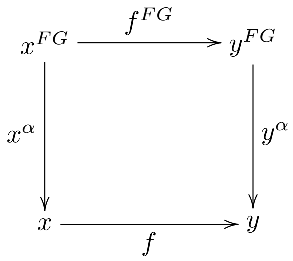
This equality can also be calculated perfectly algebraically. Start with the naturality condition for β as applied to f, and substitute the previously calculated equation, then apply the naturality condition for β as applied to $\tilde{f}^{F}$:

$$\begin{aligned}
x^{F\beta} f &= f^{GF} y^{F\beta} \\
&= \tilde{f}^{FGF} y^{F\beta} \\
&= x^{F\beta} \tilde{f}^{F}; \\
f &= \tilde{f}^{F}.
\end{aligned}$$

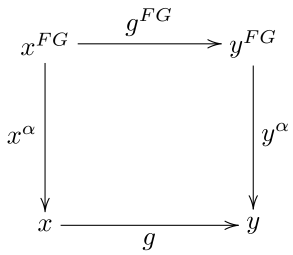
Faithfulness

In this section, I prove that equivalences are faithful. So let C, D, F (with G, α, and β), x, and y be as above, and let $f, g\colon x \to y$ be morphisms in C. Suppose that $f^{F} = g^{F}\colon x^{F} \to y^{F}$ in D. Then $f^{FG} = g^{FG}\colon x^{FG} \to y^{FG}$ also.

Now look at the naturality diagrams for α as applied to f and g:



and:



These are identical except for the horizontal arrows. But the top arrows are equal, so the bottom arrows must also be equal. Therefore, $f = g$, and the functor F is faithful.

Again, this can be done by manipulating equations:

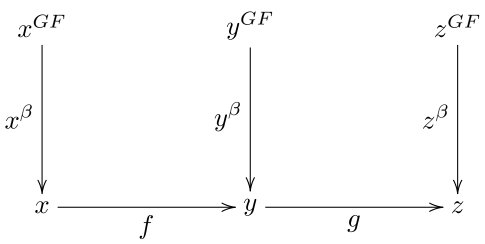
$$\begin{aligned} x^\alpha f &= f^{FG} y^\alpha \\ &= g^{FG} y^\alpha \\ &= x^\alpha g; \\ f &= g. \end{aligned}$$

Equivalence

In this section, I prove that an essentially surjective, full, faithful functor is an equivalence. So let C and D be categories, and let F be a functor from C to D. Suppose that F is essentially surjective, full, and faithful. I wish to construct a weak inverse G of F, with associated natural isomorphisms α and β.

First, let me define G on objects. If x is an object of D, then by the essential surjectivity of F, there is an object $\tilde{x}$ of C such that $\tilde{x}^F \cong x$. So let x^G be this $\tilde{x}$. As a bonus, I see that β is already staring at me: let $x^\beta\colon x^{GF} \to x$ be the guaranteed isomorphism from $\tilde{x}^F$ to x. Next, let me define G on morphisms. If x and y are objects of D and $f\colon x \to y$ is a morphism in D, then $g := x^\beta f y^{-\beta}$ is a morphism from x^{GF} to y^{GF}. Since F is full, I get a corresponding morphism $\tilde{g}$ from x^G to y^G in C, such that $\tilde{g}^F = g$. Let f^G be this $\tilde{g}$. Finally, I must define α. If x is an object in C, then $f := x^{F\beta}$ is a morphism in D from x^{FGF} to x^F. Since F is full, there must be a morphism $\tilde{f}\colon x^{FG} \to x$; let x^α be this $\tilde{f}$.

Now, the hard part is proving that the above really defines a functor and natural transformations. First, let me check that G preserves identities. Given an object x of D, what is 1_x^G? Well, if $g := x^\beta 1_x x^{-\beta} = 1_{x^{GF}} = 1^F_{x^G}$, then 1^G_x is a lifting of g such that $1^{GF}_x = g = 1^F_{x^G}$. Since F is faithful, it follows that $1^G_x = 1_{x^G}$. Thus, G preserves identities. Now let me check that G preserves composition. Given objects x, y, and z of D and morphisms $f\colon x \to y$ and $g\colon y \to z$, I see that f^G is a lifting of $x^\beta f y^{-\beta}$ and g^G is a lifting of $y^\beta g z^{-\beta}$, while $(fg)^G$ is a lifting of $x^\beta f g z^{-\beta}$. These will be easier to think about with a picture:

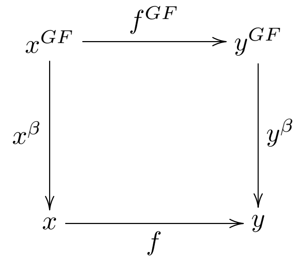


Then the definition of f^G shows that f^{GF} runs along the left half of the top row; similarly, g^{GF} runs along the right half. But the definition of $(fg)^G$ shows that the entire top row can be filled by the single arrow $(fg)^{GF}\colon x^{GF} \to y^{GF}$. Thus $(fg)^{GF} = f^{GF}g^{GF} = (f^G g^G)^F$. I can also get this equation purely algebraically:

$$\begin{aligned}(f^G g^G)^F &= f^{GF} g^{GF} \\ &= x^\beta f y^{-\beta} y^\beta g z^{-\beta} \\ &= x^\beta f g z^{-\beta} \\ &= (fg)^{GF}.\end{aligned}$$

Either way, since F is faithful, it follows that $(fg)^G = f^G g^G$. Therefore, G is indeed a functor.

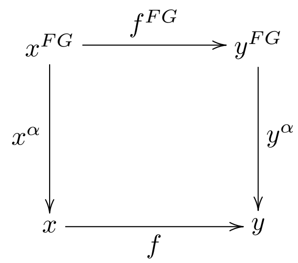
Next, let me check that β is a natural transformation. That is, if $f\colon x \to y$ is a morphism in D, then I want to prove that this square commutes:



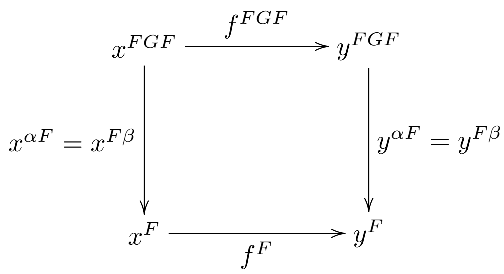
But this is true by the very *defintion* of f^G. In equations,

$$\begin{aligned} f^{GF} &:= x^\beta f y^{-\beta}; \\ f^{GF} y^\beta &= x^\beta f. \end{aligned}$$

Finally, let me check that α is a natural transformation. That is, if $f\colon x \to y$ is a morphism in C, then I want to prove that this square commutes:



Whether or not this commutes, it still exists, so I'll apply F to it:



Here I've written in the definition of α in terms of β. But now this is simply the commutative square for β applied to f^F. Thus the original square also commutes, since F is faithful. This also can be done with

1dimensional algebra:

$$\begin{aligned} f^{FGF} &= x^{F\beta} f^F y^{-F\beta}; \\ f^{FGF} y^{F\beta} &= x^{F\beta} f^F \\ f^{FGF} y^{\alpha F} &= x^{\alpha F} f^F; \\ f^{FG} y^{\alpha} &= x^{\alpha} f. \end{aligned}$$

Therefore, F is an equivalence.

Constructivity

In this section, I examine to what extent the previous section should be acceptable to a constructivist; in particular, does it rely nonconstructively on the axiom of choice?

At first glance, this would seem to be true for the definition of G on objects. The essential surjectivity of F guarantees only the existence of *some* object $\tilde{x}$ in the essential preimage of any object x of D; it doesn't tell us how to choose *which* object to let x^G be. And indeed, if one were to formalise this proof in Zermelo Fraenkel set theory ZF (assuming that C and D are small categories), then the axiom of choice would be required here.

In fact, the axiom of choice is *equivalent*, in ZF (or even the weaker constructive Zermelo set theory CZ) to the theorem that every essentially surjective, full, faithful functor between small categories has a weak inverse. To see this, let $\mathcal{S}$ be a collection of occupied sets. Define a category C such that $\operatorname{Ob} C = \biguplus \mathcal{S}$; that is, an object of C is a pair (s, S) such that $s \in S \in \mathcal{S}$. Given objects $x := (s, S)$ and $y := (t, T)$ of C, let there be a morphism from x to y iff $S = T$, and let this morphism be unique. Also, define a category D such that $\operatorname{Ob} D = \mathcal{S}$; let the morphisms of D be only identity morphisms. Finally, define a functor $F\colon C \to D$ by letting $(s, S)^F$ be S. Then F is clearly full and faithful; it's also essentially surjective (in fact surjective) since each set S is occupied. If a weak inverse G exists, then S^G is an element of S, so G (when restricted to objects) is a choice function for $\mathcal{S}$.

I restricted attention above to small categories, so that I could discuss set theory; the situation is even worse for large categories. If we work in a formalism allowing for large classes, then the previous paragraph can be applied when $\mathcal{S}$ is the large class of *all* occupied small sets. The result is a *global* choice function; a large function on the class of occupied small sets that maps, once and for all, each occupied small set to one of its elements. This class-theoretic axiom of choice is even stronger than the purely set-theoretic axiom.

Yet as Peter Aczel wrote, '[t]he axiom of choice has an ambiguous status in constructive matematics'. It has long been known that CZ + AC implies the law of the excluded middle (since one can force the choice function to choose between a proposition and its negation), so the set-theoretic axiom of choice cannot be accepted in constructive set theory. On the other hand, any constructive proof that every member of $\mathcal{S}$ is occupied must give an algorithm for turning a construction of any $S \in \mathcal{S}$ into a construction of some $s \in S$. If one identifies a set with the set of constructions of its elements (as Per Martin-Löf encourages), then this algorithm describes a choice function.

Constructivist philosophers today know that the resolution of this conflict lies in understanding the role of *equality* in set theory. Two quite different constructions of members of $\mathcal{S}$ may turn out to describe equal sets; yet when applying the proof that every member of $\mathcal{S}$ is occupied to these constructions, the resulting elements of the equal sets may be quite different. Thus we have a choice *operation* (also called *prefunction*, or *function of complete presentations*), but not a choice *function*, because an operation is a function only if it preserves equality.

This may become clearer with an example; let me use the most dangerous example, the $\mathcal{S}$ whose choice function proves the law of the excluded middle. Actually, we have a seperate $\mathcal{S}$ for each proposition P; if $\mathcal{S}$ has a choice function, then we can prove $P \vee \neg P$. Now, $\mathcal{S}$ will be a collection of subsets of $\{0, 1\}$, and it will consist of two (possibly equal) subsets, S and T. To define S and T, I must explain whether 0 and 1 belong to them. So, let $0 \in S$ and $1 \in T$ regardless; but let $0 \in T$ and $1 \in S$ each iff P is true. Now, to prove that S and T are occupied, you can ignore P and just note that $0 \in S$ and $1 \in T$. So a choice operation for $\mathcal{S}$ maps S to 0 and T to 1. But this will not be a function if $S = T$! This is relevant to P, because $S = T$ iff P is true. In other words, if this choice operation is a function, then P is false. Now, any of the other operations from $\mathcal{S}$ to $\{0,1\}$ require P to be true if they are to be choice operations. So if you insist that $\mathcal{S}$ has a choice *function*, then either P is false or it is true. But if you require only that $\mathcal{S}$ have a choice *operation*, then you can draw no conclusion about P.

Thus practising constructivists may use the axiom of choice, so long as they verify that the choice operation is a function by checking that it preserves equality. My construction in the previous section is perfectly acceptable to a constructivist so long as it's understood that I'm only defining operations; to define functions, something more is required. But what does it mean to preserve equality in category theory? Equality between objects has no meaning! Only equality between morphisms has meaning, and this does indeed have to be checked. But equality between objects must be replaced by isomorphism, and it's the very definition of G on morphisms that shows that the definition of G on objects is an operation that preserves isomorphism. In this sense, functoriality of an operation on objects (being able to extend the operation appropriately to morphisms) is a categorification of functionality of an operation on elements (being able to prove that the function preserves equality).

Let's examine the previous section carefully in light of this. I defined G on objects, as an operation from $\operatorname{Ob} D$ to $\operatorname{Ob} C$. Then I defined β, as an operation from $\operatorname{Ob} D$ to $\operatorname{Mor} C$. Then I defined G on morphisms, as an operation from $\operatorname{Mor} D$ to $\operatorname{Mor} C$. Finally, I defined α, as an operation from $\operatorname{Ob} C$ to $\operatorname{Mor} D$. Since none of these classes of objects or morphisms has any absolute notion of equality, to what extent is it reasonable to demand that these operations be functions? They *should* be functions to that extent; and if I haven't proved it yet, then I should do so now. But it makes no sense to expect them to be functions in any absolute sense, so I certainly don't have to prove that.

Let's start with G on morphisms. Given objects x and y in D, it makes sense to say that two morphisms from x to y are equal; similarly, it makes sense to say that two morphisms from x^G to y^G are equal. Thus, I should prove that G defines a function from $\operatorname{Hom}(x,y)$ to $\operatorname{Hom}(x^G,y^G)$. That is, if $f = g\colon x \to y$, then $f^G = f^G\colon x^G \to y^G$. And this is true, because F is faithful. (Thus the faithfulness of F is needed in this paragraph!) Next, what does it mean to say that α or β preserves equality? Taking β for definiteness, I can't expect to say that $x^\beta = y^\beta$ if $x = y$, because the latter equation has no meaning. Nor can I expect to say that $x^\beta = y^\beta$ if $x \cong y$, because then x^β and y^β may not even have the same domain and codomain. All that I can expect is that, given an isomorphism $f\colon x \xrightarrow{\sim} y$, I should find that x^β and y^β are related through f^{GF} and f. And they are — this is what the naturality square of β says! (The same idea holds for α.) Finally, I've already dealt with G on objects. It's meaningless to consider whether $x = y$, but if $x \cong y$, then I should have $x^G \cong y^G$. And this is true; if $f\colon x \xrightarrow{\sim} y$, then $f^G\colon x^G \xrightarrow{\sim} y^G$.

Why doesn't this prove the axiom of choice, using the argument from the beginning of this section? Recall that $\mathcal{S}$ was a collection of occupied sets (occupied subsets of a given universe, to be careful), and I defined a functor whose weak inverse G, when restricted to objects, defined a choice function on $\mathcal{S}$. Now, this is certainly a fine choice *operation* on $\mathcal{S}$, but is it a function? The only sense in which it's a function is that if $S \cong T$, then $S^G \cong T^G$. This is perfectly true, since $S \cong T$ iff $S = T$ and $S^G \cong T^G$ iff $S = T$ as well. But this does not mean that $S^G = T^G$! Even though that statement has meaning, because that the elements of $\mathcal{S}$ were subsets from a given universe, it does not actually follow from the functoriality of G.

Notice how the 'functions' of category theory — functors and natural transformations — are, despite their apparently complicated definitions, really just the natural notions of functionality in a situation where equality of objects is replaced by isomorphism. That is, if one has an operation mapping objects of one category to objects of another category, then the closest thing to requiring this operation to be a function is to extend the operation to isomorphisms, making it a functor, at least on the underlying groupoids. (Actually, a bit more insight is necessary to realise that a good notion of functor should also preserve composition.) Similarly, if one has an operation mapping objects of one category to morphisms of another category, then the closest thing to requiring this operation to be a function is to identify the functors involved and make the naturality square commute. Before category theory, mathematicians were working with such operations commonly, but since most mathematicians believed in a global notion of equality, they didn't realise that they weren't functions. A proper appreciation of the relativity of equality and the distinction between operation and function leads naturally to the full development of groupoid theory (after which the generalisation to category theory is obvious). Conversely, a proper appreciation for the role of the axiom of choice in constructive mathematics confirms that there is nothing nonconstructive about category theory.