

Categorified Inner Products, Revisited

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Recall that the **field operator**

$$\phi: \mathbb{C}[z] \rightarrow \mathbb{C}[z]$$

is defined by

$$\phi = a + a^*$$

where a and a^* are the **annihilation** and **creation** operators, respectively, given by

$$af = \frac{df}{dz}, \quad a^*f = zf$$

for all $f \in \mathbb{C}[z]$. Recall also that $\mathbb{C}[z]$ has a unique inner product such that

$$\langle 1, 1 \rangle = 1$$

and

$$\langle a^*f, g \rangle = \langle f, ag \rangle.$$

This inner product is given by:

$$\langle z^k, z^\ell \rangle = \delta_{ki} k!$$

1. Work out the inner product $\langle 1, \phi^4 1 \rangle$ ‘by hand’, using only the facts above.

Hint: I want you to consider this sum of 2^4 terms:

$$\langle 1, \phi^4 1 \rangle = \langle 1, (a + a^*)(a + a^*)(a + a^*)(a + a^*)1 \rangle.$$

Most of these are zero for various reasons. Figure out these reasons, figure out which terms aren't zero, figure out what they equal, and add them up.

You should already know the final answer to this question, since we've seen in class that for $n > 0$,

$$\langle 1, \phi^n 1 \rangle = \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & n \text{ even,} \end{cases} \quad (1)$$

*and in the homework on **Wick Powers** we've seen one reason why this is important. So, the point here is to work out the answer carefully from scratch, without using stuff operators or Feynman diagrams. And then....*

2. Calculate the groupoid

$$\langle 1, \Phi^4 1 \rangle$$

where 1 stands for the stuff type ‘being the empty set’ and $\Phi = A + A^*$ is the stuff operator described in class.

3. Prove that

$$|\langle 1, \Phi^4 1 \rangle| = \langle 1, \phi^4 1 \rangle$$

by showing how each term in the sum you got for $\langle 1, \phi^4 1 \rangle$ in Problem 1 corresponds to a specific collection of objects in the groupoid $\langle 1, \Phi^4 1 \rangle$.

Here's the point: categorification turns numbers into bunches of objects. *More precisely, when we categorify a sum of numbers $x = \sum x_i$ and get a sum of groupoids $X = \sum X_i$ with $|X_i| = x_i$, each term x_i corresponds to a specific groupoid X_i . Since X_i is a full subcategory of X , it corresponds to a specific collection of objects of X (together with all the morphisms between these). So, each term in the sum for $\langle 1, \phi^n 1 \rangle$ corresponds to a specific collection of objects in $\langle 1, \Phi^n 1 \rangle$. Pondering this should help you understand what's really going on when we use Feynman diagrams to compute inner products.*

4. Compute

$$\langle z^2, \frac{\phi^3}{3!} \frac{\phi^3}{3!} z^2 \rangle$$

without using stuff operators or Feynman diagrams, but with the help of equation (1) if you like.

5. Compute the groupoid

$$\langle Z^2, (\Phi^3 // 3!) (\Phi^3 // 3!) Z^2 \rangle$$

where Z^2 is the stuff type 'being a totally ordered 2-element set' and $\Phi^n // n!$ is the stuff operator described in terms of Feynman diagrams in class.

6. Compute the groupoid cardinality

$$|\langle Z^2, (\Phi^3 // 3!) (\Phi^3 // 3!) Z^2 \rangle|$$

and compare this to your answer to Problem 4.