DEREK WISE

22 May 2004

QUANTUM GRAVITY SEMINAR

Categorified Inner Products, Revisited

1) $\varphi^4 = (a+a^*)^4$

$$
\begin{array}{ll}
= a^4 & \\
+a^*a^3 + aa^*a^2 + a^2a^*a & \\
+a^{*2}a^2 + a^*aa^*a + aa^{*2}a & \\
+a^{*3}a &
\end{array}
\Bigg\}
$$

Each of these terms is of the form xa, so $\langle 1, xa1\rangle = \langle 1, x0\rangle = 0$ and these don't contribute to $\langle 1, \varphi^4 1\rangle$

$+a^*a^2a^* + a^3a^*$ $\}$ $\langle 1, xa^2a^*1\rangle = \langle 1, xa^2z\rangle = \langle 1, xa1\rangle = 0$

$$
\begin{array}{l}
+a^{*4} \\
+aa^{*3} \\
+a^*aa^{*2} \\
+a^{*2}aa^*
\end{array}
\Bigg\}
$$

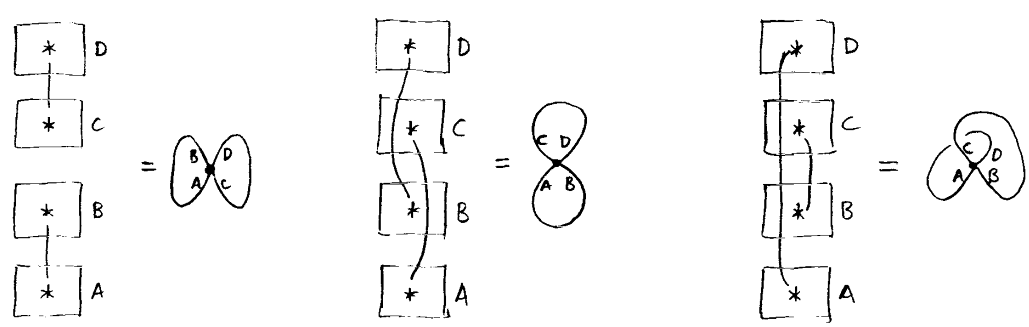
These all give multiples of z^n with $n \geq 1$, and $\langle 1, z^n\rangle = 0 \quad \forall n \geq 1$ so these don't contribute either.

$+a^2a^{*2} \quad + aa^*aa^*$ $\}$ Our only contributors!

So
$$
\begin{aligned}
\langle 1, \varphi^4 1\rangle &= \langle 1, a^2a^{*2}1\rangle + \langle 1, aa^*aa^*1\rangle \\
&= \langle 1, a^2z^2\rangle + \langle 1, aa^*1\rangle \\
&= \langle 1, 2\rangle + \langle 1, 1\rangle \\
&= 3
\end{aligned}
$$

which is consistent with our result $\langle 1, \varphi^4 1\rangle = (4-1)!! = 3\cdot 1 = 3$.

2) The objects of the groupoid $\langle 1, \Phi^4 1\rangle$ look like

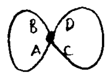


$\cap$ $\langle 1, AA^*AA^*1\rangle$ $\qquad \cong \quad \cong$ $\langle 1, A^2A^{*2}1\rangle$

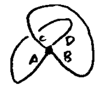
and there are only identity morphisms, so

$$\langle 1, \Phi^4 1\rangle \simeq 3,$$

the 3-elt set.

3) The object 

is an object of the groupoid $\langle 1, AA^*AA^*1\rangle$ while the objects 

& 

are in $\langle 1, A^2A^{*2}1\rangle$. All of the other summands of $\langle 1, \Phi^4 1\rangle$ are the empty groupoid.

The table from part (1) becomes:

$\Phi^4 = (A + A^*)^4$

$= A^4$
$+ A^*A^3 + AA^*A^2 + A^2A^*A$
$+ A^{*2}A^2 + A^*AA^*A + AA^{*2}A$
$+ A^{*3}A$

} Each of these stuff operators is of the form XA, and $A1$ is the empty groupoid, so $\langle 1, XA1\rangle$ is also the empty groupoid since there are no isomorphisms $1 \longrightarrow 0$.

$+ A^*A^2A^* + A^3A^*$

} $A^*1 \simeq Z$ and A^2Z is the empty groupoid. So $\langle 1, XA^2A^*1\rangle$ is also empty.

$+ A^{*4}$
$+ AA^{*3}$
$+ A^*AA^{*2}$
$+ A^{*2}AA^*$

} $A^{*4}1 \simeq Z^4$
$AA^{*3}1 \simeq 3Z^2$
$A^*AA^{*2}1 \simeq 2Z^2$
$A^{*2}AA^*1 \simeq Z^2$

so these all give $\langle 1, X1\rangle$ the empty groupoid.

$+ A^2A^{*2} + AA^*AA^*$

} Our only contributors to $\langle 1, \Phi^4 1\rangle$.

4) $\frac{\varphi^3}{3!} z^2 = \frac{1}{3!}(a+a^*)^3 z^2$

$$= \frac{1}{3!}(a+a^*)^2(a+a^*) z^2$$

$$= \frac{1}{3!}(a+a^*)^2(2z+z^3)$$

$$= \frac{1}{3!}(a+a^*)(2+2z^2+3z^2+z^4)$$

$$= \frac{1}{3!}(a+a^*)(2+5z^2+z^4)$$

$$= \frac{1}{3!}(2z+10z+5z^3+4z^3+z^5)$$

$$= \frac{1}{3!}(12z+9z^3+z^5)$$

So, since $\varphi = \varphi^*$, we get:

$$\left\langle z^2, \frac{\varphi^3}{3!}\frac{\varphi^3}{3!} z^2 \right\rangle = \left\langle \frac{\varphi^3}{3!} z^2, \frac{\varphi^3}{3!} z^2 \right\rangle$$

$$= \frac{1}{(3!)^2}\left\langle 12z+9z^3+z^5, 12z+9z^3+z^5 \right\rangle$$

$$= \frac{1}{(3!)^2}\cdot\left(12^2\langle z,z\rangle + 9^2\langle z^3,z^3\rangle + \langle z^5,z^5\rangle\right)$$

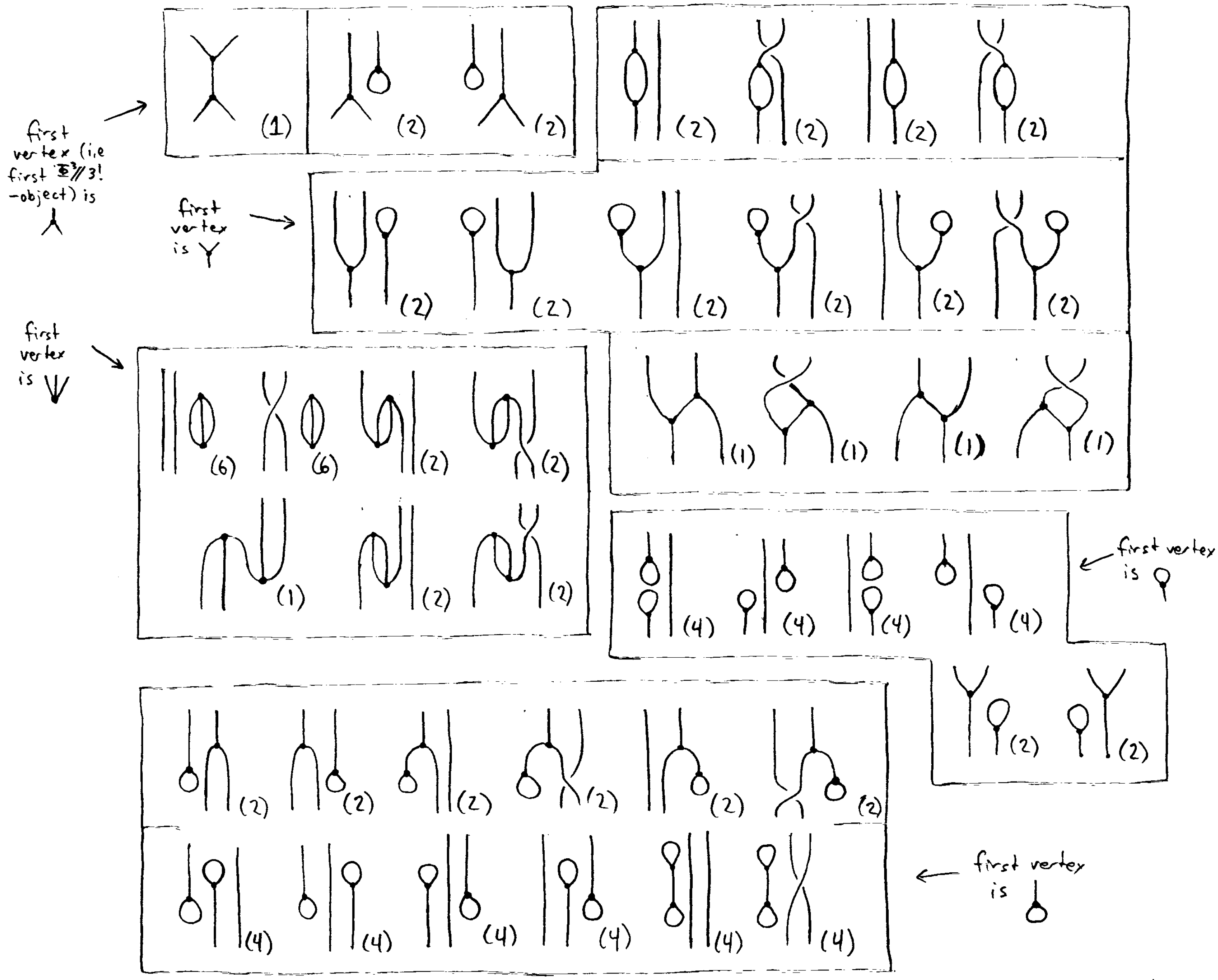
$\langle z^n, z^m\rangle = \delta_{nm} n!$

$$= \frac{1}{(3!)^2}\cdot(144 + 81\cdot 3! + 5!)$$

$$= \frac{750}{6^2}$$

$$= \frac{125}{6}$$

5) An object of $\langle Z^2, \frac{\Phi^3}{3!} \frac{\Phi^3}{3!} Z^2 \rangle$ is an object of Z^2, two objects of $\frac{\Phi^3}{3!}$ and another object of Z^2, all connected via the isomorphisms in the definition of weak pullback. In other words, they are Feynman diagrams with two trivalent vertices, two (ordered) incoming particles and two (ordered) outgoing particles. Here are all 42 of the objects (in a skeletal version) along with the sizes of their automorphism groups



(there are none whose first vertex is $\wedge$)

6) So, counting up the number of symmetry groups of each size, we get:

$$\left|\langle Z^2, \frac{\Phi^3}{3!} \frac{\Phi^3}{3!} Z^2 \rangle\right| = 6\left(\frac{1}{1}\right) + 24\left(\frac{1}{2}\right) + 10\left(\frac{1}{4}\right) + 2\left(\frac{1}{6}\right) = \boxed{\frac{125}{6}} = \langle z^2, \frac{\varphi^3}{3!} \frac{\varphi^3}{3!} z^2 \rangle$$

just as we found before.