Being an Octopus[1]

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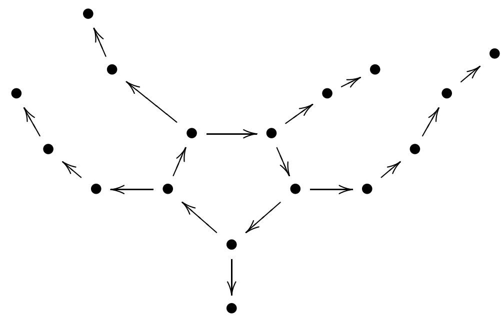
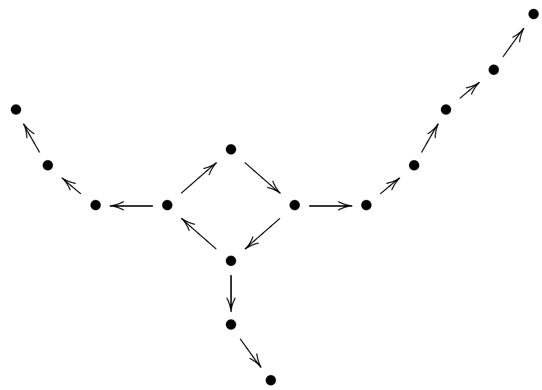
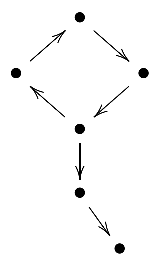
In their book **Combinatorial Species and Tree-Like Structures***, Bergeron, Labelle and Leroux discuss a structure type* Oct*, called* **"being an octopus"***. Instead of defining it, I'd like you to guess it from some hints, and then work out its generating function. Here is a way to put an octopus structure on a 17-element set:*



Here is a way to put an octopus structure on a 14-element set:

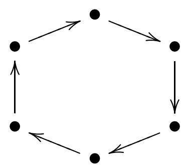


Here is one way to put an octopus structure on a 6-element set:



[1]The philosopher Heidegger wrote a book called **Being and Time**. The philosopher Sartre wrote a book called **Being and Nothingness**. Once I too wanted to be a philosopher; here is my pathetic attempt to follow in their footsteps.

and here is another:



Recall that given stuff types F and G, there is a stuff type $F \circ G$ such that to put $F \circ G$-stuff on a finite set S, we write S as a disjoint union $S_1 + \cdots + S_n$, put F-stuff on the set $\{1, \ldots, n\}$, and put G-stuff on each of the sets S_i. We have

$$|F \circ G| = |F| \circ |G|$$

whenever either side is a well-defined formal power series.

1. Find structure types F and G such that $\mathrm{Oct} \cong F \circ G$.

An octopus is a cyclically ordered family of octopus arms, where an octopus arm is a nonempty totally ordered set. Thus F is the structure type of being a cyclically ordered set, while G is the structure type of being an octopus arm, in other words of being a nonempty totally ordered set. Note that placing a Gstructure on an $n$set consists of choosing a point at the head of the list (or in this case, the head of the octupus) and totally ordering the reminder. I can express these in terms of the homework on cyclic orderings: $F = C$, while $G = ZL = Z\frac{D}{DZ}C$.

2. Work out $|F|$ and $|G|$ and use this to work out $|\mathrm{Oct}|$.

I begin with $|C| = -\ln(1-z)$. Then $|F| = |C| = -\ln(1-z)$, while $|G| = z\frac{d}{dz}|C| = z/(1-z)$. Therefore, $|\mathrm{Oct}| = |F \circ G| = |F| \circ |G| = -\ln(1 - z/(1-z))$.

3. Find a simple explicit formula for the nth coefficient of the formal power series $|\mathrm{Oct}|(z)$. Use this to count the number of ways to put an octopus structure on an n-element set.

I'm not clever enough to think of a formula, but I can calculate the first few numbers. The number of octopus structures on an $n$set is, in general, equal to $|\mathrm{Oct}|^{(n)}(0)$, and I get $|\mathrm{Oct}|(0) = 0$, $|\mathrm{Oct}|'(0) = 1$, $|\mathrm{Oct}|''(0) = 3$, $|\mathrm{Oct}|'''(0) = 14$, and $|\mathrm{Oct}|''''(0) = 90$. (I've also drawn and counted the first 18 octopi.) The On-Line Encyclopedia of Integer Sequences suggests that this is sequence A029767, with the formula $(n-1)!\,(2^n - 1)$. This formula can't be correct for $n = 0$, so I hypothesise that it's correct for all other values.

In fact, given this hint, I can analyse the structure Oct in another way, proving the formula's correctness at the same time. First, $2^n - 1$ is the number of ways to choose a nonempty subset of an $n$set. Surely, this subset should be the head of the octopus. Next, $(n-1)! = |C|^{(n)}(0)$ is the number of ways to cyclically order an $n$set (except for $n = 0$, of course). Now, the *arms* are supposed to be cyclically orderd, but is the entire set cyclically ordered? Yes, it is! Simply go along each arm in order, then go on to the next arm. Conversely, if you start with a cyclically ordered set, then all that you need to do to make this into an octopus is to choose a nonempty subset to comprise the head. Another way to look at this is that a cyclically ordered family of totally ordered sets consists of a cyclically ordered set that's been partitioned into nonempty *contiguous* parts.

Note that the power series for $|\mathrm{Oct}|$ is

$$|\mathrm{Oct}| = \sum_{n=1}^{\infty} \frac{(n-1)!\,(2^n - 1)}{n!} z^n = \sum_{n=1}^{\infty} \frac{2^n - 1}{n} z^n.$$

(By the way, while I was checking this series with Mathematica, I accidentally discovered that $(2^n - 1)/n = \ln 2$ when $n = 0$. This is true as a limit in a continuous variable n, but I don't know why Mathematica took that limit when summing the discrete series. That would certainly be unwise behavior for an expression involving 0^n, which is discontinuous as $n \to 0$, even though it's well defined when $n = 0$. Don't we flunk calculus students for this sort of thing? In actual tests, Mathematica refuses to even attempt $\sum_{n=1}^{\infty} 0^n$, even though *this* is absolutely trivial! And have you ever noticed that Mathematica never tells you the interval of convergence? Conclusion: Mathematica ought to flunk calculus.)