Perturbation Theory

John C. Baez, April 27, 2004

In physics we often try to study the dynamics of a complicated system by thinking of it as a slightly modified version of some simpler system — preferably one where we can compute everything in 'closed form'. We then use the simpler system as a starting point for studying the more complicated one. This idea is called **perturbation theory**.

In quantum theory we sometimes do this as follows. Suppose we have a system with Hilbert space $\mathbf{H}$ *whose Hamiltonian is some self-adjoint operator* H *on* $\mathbf{H}$. *States are described by unit vectors in* $\mathbf{H}$ *that depend on time. These evolve according to* **Schrödinger's equation***:*

$$\frac{d\psi(t)}{dt} = -iH\psi(t)$$

It's easy to write down the unique solution to this equation with $\psi(0)$ *equal to a given state* $\psi \in \mathbf{H}$*:*

$$\psi(t) = e^{-itH}\psi$$

where

$$e^{-itH}\psi = \sum_{n=0}^{\infty} \frac{(-itH)^n}{n!}\psi.$$

(If H *is any bounded operator, the right-hand side converges in the norm topology on* $\mathbf{H}$ *for all vectors* ψ. *If* H *is an unbounded self-adjoint operator, it converges for a dense set of vectors called* **entire vectors***; we can then define* $e^{-itH}\psi$ *for other vectors by applying* e^{-itH} *to a sequence of entire vectors that converges to* ψ *and taking the limit. To avoid subtleties like this and focus attention on the basic ideas, let's assume in Problems 1 and 2 that all operators under discussion are* bounded. *Unfortunately this does not hold in the really interesting examples, like in Problem 3. So, things get more technical — but the basic ideas are still relevant.)*

Even though the solution to Schrödinger's equation is easy to write down, when H *is complicated it's hard to actually* calculate $e^{-itH}\psi$. *To deal with this, we often try to write*

$$H = H_0 + V$$

where H_0 *and* V *are self-adjoint operators. We try to do this so that* $e^{-itH_0}\psi$ *is easy to calculate and* V *is small. Then we write* $e^{-itH}\psi$ *as an infinite sum where the zeroth-order term is just* $e^{-itH_0}\psi$, *while the* n*th-order term involves* n *factors of* V.

In physics jargon we call H_0 *the* **free Hamiltonian** *and* V *the* **interaction Hamiltonian**. *The power series for* $e^{-itH}\psi$ *is called a* **perturbation series**. *With no further ado, here is how it actually looks:*

$$\psi(t) = \tag{1}$$

$$\sum_{n=0}^{\infty} \int_{0 \le t_1 \le \cdots \le t_n \le t} (-i)^n e^{-i(t-t_n)H_0} V e^{-i(t_n - t_{n-1})H_0} V \cdots e^{-i(t_2-t_1)H_0} V e^{-it_1 H_0} \; \psi \;\; dt_1 dt_2 \cdots dt_n.$$

1. Show that this equation is true.

Hint: Here's one way. The basic theorem on ordinary differential equations — Picard's theorem — assures us that when H is bounded, Schrödinger's equation

$$\frac{d\psi(t)}{dt} = -iH\psi(t)$$

has a unique solution with the initial conditions

$$\psi(0) = \psi.$$

So, it suffices to show that if we define $\psi(t)$ using (1), it satisfies Schrödinger's equation and these initial conditions. I'll be satisfied if you check this at the physicist's level of rigor — namely, by taking (1) and performing plausible manipulations without checking the analyst's fine print that guarantees they're allowed. It's actually easy to check this fine print when H_0 and V are bounded — but that's not the point of this exercise!

We can make the meaning of the perturbation series clearer if we work in the **interaction representation**. *In this approach, we to 'factor out' the effect on time evolution due to the free Hamiltonian. To do this, we focus attention on*

$$\psi_{\mathrm{int}}(t) = e^{itH_0}\psi(t)$$

instead of $\psi(t)$. Similarly, we focus attention on

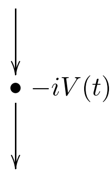
$$V(t) = e^{itH_0}Ve^{-itH_0}$$

instead of V. Since we're assuming e^{itH_0} is easy to compute, a formula $\psi_{\mathrm{int}}(t)$ is just as good as a formula for $\psi(t)$. And here it is:

2. Starting with equation (1), show that

$$\psi_{\mathrm{int}}(t) = \sum_{n=0}^{\infty} \int_{0 \le t_1 \le \cdots \le t_n \le t} (-i)^n V(t_n) \cdots V(t_1)\psi \, dt_1 \cdots dt_n. \qquad (2)$$

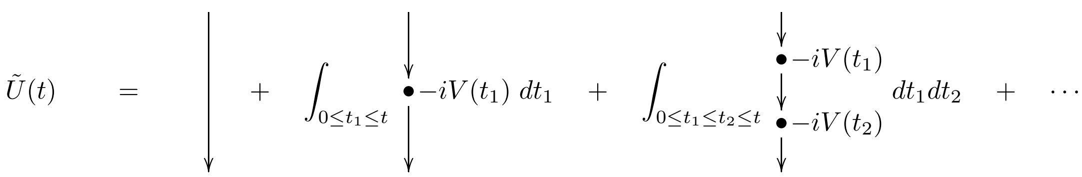
Formula (2) will eventually lead us to Feynman diagrams if we start drawing pictures like this:



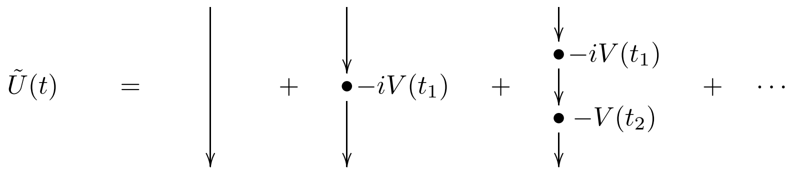
to stand for the operator $-iV(t)$. We think of this as a picture of the system evolving in a boring way according to the free Hamiltonian except for an 'interaction' that occurs at time t. If we use the usual trick for drawing composition of linear operators, formula (2) says that the time evolution in the interaction representation:

$$\begin{array}{rccc} \tilde{U}(t)\colon & \mathbf{H} & \to & \mathbf{H} \\ & \psi & \mapsto & \tilde{\psi}(t) \end{array}$$

is given as follows:



Usually when people draw these diagrams, the integrals over the times at which interactions occur are left implicit, with the vertical ordering of the dots serving to remind us that $t_1 \leq \cdots \leq t_n$. In this simplified notation, we have:



Now let's do an example! Let's see what happens when we perturb the harmonic oscillator. So, take our Hilbert space to be the Fock space $\mathbb{K}$. Take H_0 to be the **harmonic oscillator Hamiltonian** *with the ground state energy subtracted off:*

$$H_0 = \frac{1}{2}(p^2 + q^2 - 1) = a^* a.$$

And, for simplicity, take

$$V = \lambda q,$$

where

$$q = \frac{a + a^*}{\sqrt{2}}$$

is the **position operator** *and the constant $\lambda \in \mathbb{R}$ says how strong the interaction Hamiltonian V is. (Physicists call any constant that does this sort of thing a* **coupling constant***.) This problem amounts to studying a particle on the line moving in the potential $\frac{1}{2}q^2 + \lambda q$, where the first term is the potential for the harmonic oscillator.*

To keep things simple, let's work out the amplitude for the ground state $1 \in \mathbb{K}$ to evolve to the ground state after some time t:

$$\langle 1, e^{-itH} 1 \rangle.$$

(We'd take the absolute value of this amplitude and square it to get the probability *that this process occurs.) Let's do this to second order in perturbation theory. So:*

3. Calculate the right-hand side of

$$\begin{aligned} \langle 1, e^{-itH} 1 \rangle \quad \approx \quad & \langle 1, e^{-itH_0} 1 \rangle \quad + \\ & (-i) \int_{0 \leq t_1 \leq t} \langle 1, e^{-i(t-t_1)H_0} V e^{-it_1 H_0} 1 \rangle \; dt_1 \quad + \\ & (-i)^2 \int_{0 \leq t_1 \leq t_2 \leq t} \langle 1, e^{-i(t-t_2)H_0} V e^{-i(t_2-t_1)H_0} V e^{-it_1 H_0} 1 \rangle \; dt_1 dt_2 \end{aligned}$$

Your answer should be a completely explicit function of λ and t.

Hint: you'll want to review how a, a^* and thus H_0 and q act on the basis vectors $z^n \in \mathbb{K}$.