**Math 260: Perturbation theory**

Miguel Carrión Álvarez

**1.** *The perturbation series.*

Starting with

$$\psi(t) = \sum_{n\geq 0} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} e^{-i(t-t_n)H_0}(-iV)e^{-i(t_n-t_{n-1})H_0} \cdots e^{-i(t_2-t_1)H_0}(-iV)e^{-it_1H_0}\psi dt_1 \cdots dt_{n-1}\, dt_n$$

we get

$$\begin{aligned}\partial_t\psi(t) =& \sum_{n\geq 1} \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_2} (-iV)e^{-i(t-t_{n-1})H_0} \cdots e^{-i(t_2-t_1)H_0}(-iV)e^{-it_1H_0}\psi dt_1 \cdots dt_{n-2}\, dt_{n-1} + \\ &+ \sum_{n\geq 1} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} (-iH_0)e^{-i(t-t_n)H_0} \cdots e^{-i(t_2-t_1)H_0}(-iV)e^{-it_1H_0}\psi dt_1 \cdots dt_{n-1}\, dt_n \\ =& - i(V + H_0)\psi(t)\end{aligned}$$

which is Schrödinger's equation.

**2.** *The perturbation series in the interaction picture.*

$$\begin{aligned}\psi(t) =& \sum_{n\geq 0} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} e^{-i(t-t_n)H_0}(-iV)e^{-i(t_n-t_{n-1})H_0} \cdots e^{-i(t_2-t_1)H_0}(-iV)e^{-it_1H_0}\psi dt_1 \cdots dt_{n-1}\, dt_n = \\ =& e^{-itH_0} \sum_{n\geq 0} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \left[e^{it_nH_0}(-iV)e^{-it_nH_0}\right] \cdots [e^{it_1H_0}(-iV)e^{-it_1H_0}]\psi dt_1 \cdots dt_{n-1}\, dt_n\end{aligned}$$

implies

$$\psi_{\text{int}}(t) = \sum_{n\geq 0} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \left[-iV(t_n)\right] \cdots \left[-iV(t_1)\right]\psi dt_1 \cdots dt_{n-1}\, dt_n$$

**3.** *Ground state transition amplitude.*

Using the facts that $H_0 1 = 0$, and $H_0 z = z$, and the representation $a \sim \partial_z$ and $a^* \sim z$,

$$\begin{aligned}\langle 1, e^{-i(t-t_1)H_0}(-iV)e^{-it_1H_0}1\rangle =& \frac{\lambda}{i\sqrt{2}}\langle 1, e^{-i(t-t_1)H_0}(a+a^*)e^{-it_1H_0}1\rangle = \frac{\lambda}{i\sqrt{2}}\langle 1, e^{-i(t-t_1)H_0}(a+a^*)1\rangle = \\ =& \frac{\lambda}{i\sqrt{2}}\langle 1, e^{-i(t-t_1)H_0}(0+z)\rangle = \frac{\lambda}{i\sqrt{2}}\langle 1, e^{-i(t-t_1)}z\rangle = \frac{\lambda}{i\sqrt{2}}e^{-i(t-t_1)}\langle 1, z\rangle = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle 1, e^{-i(t-t_2)H_0}(-iV)e^{-i(t_2-t_1)H_0}(-iV)e^{-it_1H_0}1\rangle =& \frac{-\lambda^2}{2}\langle 1, e^{-i(t-t_2)H_0}(a+a^*)e^{-i(t_2-t_1)H_0}(a+a^*)1\rangle = \\ =& \frac{-\lambda^2}{2}\langle 1, e^{-i(t-t_2)H_0}(a+a^*)e^{-i(t_2-t_1)H_0}(0+z)\rangle = \\ =& \frac{-\lambda^2}{2}e^{-i(t_2-t_1)}\langle 1, e^{-i(t-t_2)H_0}(a+a^*)z\rangle = \\ =& \frac{-\lambda^2}{2}e^{-i(t_2-t_1)}\langle 1, e^{-i(t-t_2)H_0}(1+z^2)\rangle = \\ =& \frac{-\lambda^2}{2}e^{-i(t_2-t_1)}\langle 1, (1+e^{-i(t-t_2)2}z^2)\rangle = \\ =& \frac{-\lambda^2}{2}e^{-i(t_2-t_1)}\langle 1, 1\rangle = \frac{-\lambda^2}{2}e^{-i(t_2-t_1)}.\end{aligned}$$

It follows that

$$\langle 1, e^{-itH}1\rangle \approx 1 - \frac{\lambda^2}{2}\int_{0\le t_1\le t_2\le t} e^{-i(t_2-t_1)}dt_1\,dt_2 = 1 + i\frac{\lambda^2}{2}\int_{0\le t_2\le t}(1-e^{-it_2})\,dt_2 = 1 + \frac{\lambda^2}{2}(e^{-it}-1+it).$$

which is proportional to $(\lambda t)^2$ as $t \to 0$. In fact, since we are making an error of order $t^3$ by truncating the perturbation expansion at the second term,

$$\langle 1, e^{-itH}1\rangle = 1 + \frac{\lambda^2}{2}(e^{-it}-1+it) + O(t^3) = 1 - \frac{(\lambda t)^2}{2\cdot 2!} + O(t^3).$$