6 April 2004

Stuff Types

Review:

We say a functor $F: C \to D$ is

- essentially surjective if $\forall d \in D$ $\exists \tilde{d} \in C$ s.t. $F(\tilde{d}) \cong d$.
- full if $\forall c, c' \in C$,
$$F: \hom(c,c') \to \hom(F(c), F(c'))$$
is onto
- faithful if $\forall c, c' \in C$,
$$F: \hom(c,c') \to \hom(F(c), F(c'))$$
is one-to-one.

We say F:

- forgets nothing if it's faithful, full, & essentially surjective
- forgets properties if it's faithful & full
- forgets structure if it's faithful
- forgets stuff always

Here "forgets ____" really means "forgets at most ____,"
so each of the 4 conditions implies the next.

Def: A stuff type is a "groupoid over FinSet_0", i.e. a groupoid X and a functor

$$\begin{array}{c} X \\ \downarrow \\ \mathrm{FinSet}_0 \end{array}$$

We say "over" because the finite set $F(x)$ is like the "shadow" of $x \in X$: objects in X are finite sets equipped with extra stuff. (F forgets stuff.)

<u>Def:</u> A stuff type is a <u>structure type</u> if F forgets structure (i.e. is faithful).

<u>Warning:</u> This looks completely different from our old definition of a structure type as

$$F : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set}$$

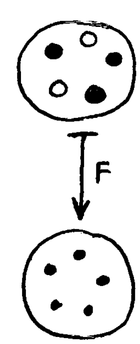
– but we'll see they're equivalent.

<u>Def:</u> A stuff type is a <u>property type</u> if F forgets properties (i.e. is faithful & full)

<u>Def:</u> A stuff type is a <u>vacuous property type</u> if F forgets nothing.

<u>Examples:</u>

1) $$\begin{array}{l} E^2 = [\text{2-colored finite sets}]_0 \\ \Big\downarrow F \quad \text{— forget the coloring} \\ E = \mathrm{FinSet}_0 \end{array}$$



(Recall: If F is a str. type & Z_0 is a groupoid, $F(Z_0)$ is the groupoid of "F-structured finite sets with elements labelled by objects of Z_0"
E^Z = "being a finite set" so E^2 = "finite sets w/ elts labelled by elts of 2 (the 2-elt set)" = "2-colored finite sets."

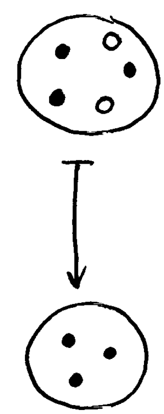
Is F :

- faithful? YES — a color preserving bijection determines a unique bijection
- full? NO — not all bijections preserve coloring
- ess. surjective? YES — any finite set can be 2-colored

So F forgets structure — it's a structure type!

Let's do a similar example: same groupoids, but different functor:

2)
$$\begin{tikzcd} E^2 = [\text{2-colored finite sets}]_0 \arrow[d, "F"'] \\ E = \mathrm{FinSet}_0 \end{tikzcd}$$
— forget the white dots



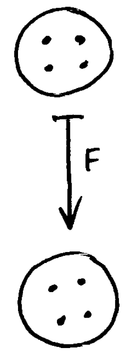
Is F :

- faithful? NO — what morphisms do to white dots gets forgotten
- full? YES — any bijection of black dots extends to a color preserving bijection
- ess. surjective? YES — any finite set is ($\cong$ to) the set of black dots in a 2-colored set.

So F forgets stuff — it's a ~~structure type~~ stuff type that's <u>not</u> a structure type.

3)
$$\begin{tikzcd} \mathrm{COSH}(1) = [\text{even sets}]_0 \arrow[d, "F"'] \\ E = \mathrm{FinSet}_0 \end{tikzcd}$$

– the obvious inclusion



(Recall: $\mathrm{COSH}(Z)$ = "being an even set")

Is F:

- faithful? YES – what morphisms do to dots is not forgotten
- full? YES – every bijection of even sets is a bijection of finite sets.
- ess. surjective? NO – E.g. the 5-elt set is not even

So F forgets properties – it's a "property type."

4) Let $\mathcal{N}$ be a skeleton of the category FinSet – a subcategory of FinSet that has one object from each isomorphism class & all morphisms between them. $\mathcal{N}$ can have objects:

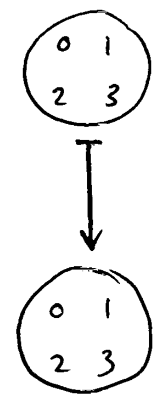
$$\begin{aligned} 0 &= \varnothing \\ 1 &= \{0\} \\ 2 &= \{0,1\} \\ 3 &= \{0,1,2\} \quad \text{etc.} \end{aligned}$$

–von Neumann's favorite n-element sets.

Let $\mathcal{N}_0$ = the groupoid with $0, 1, 2, \ldots$ as objects & bijections as morphisms (the "underlying groupoid" of $\mathcal{N}$).

Consider:

$$\begin{array}{c} \mathcal{N}_0 \\ \Big\downarrow \text{ - the obvious inclusion} \\ \mathrm{FinSet}_0 \end{array}$$



Is F:

faithful?	YES	"same hom sets"
full?	YES	
ess. surjective?	YES	- every n-elt set is $\cong$ to "the" n-elt set.

So F forgets nothing. This happens whenever F is the inclusion of a skeleton.

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8 April 2004

Stuff Types & the Harmonic Oscillator

We've seen that structure types are a categorified version of states of the quantum harmonic oscillator — the same holds for stuff types, but it works even better!

Now we'll:

1) Define a generating function $|F|(z)$ for a stuff type

$$\begin{array}{c} X \\ F \Big\downarrow \\ \mathrm{FinSet}_0 \end{array} \qquad \text{(a groupoid over } \mathrm{FinSet}_0\text{)}$$

Sometimes we'll call $\begin{array}{c} X \\ \downarrow F \\ \mathrm{FinSet}_0 \end{array}$ just "F" for short. The generating function should reduce to our previous definition when F is a structure type.

2) Define how to apply a stuff type F to a groupoid Z_0 & get a groupoid $F(Z_0)$ – the groupoid of "finite sets equipped with F-stuff & with elts. labelled by objects of Z_0," or "F-stuffed Z_0-colored finite sets," for short.

3) Check that:

$$|F(Z_0)| = |F|(|Z_0|)$$

4) Given two stuff types F & G, define a stuff type $F \circ G$ such that

$$F \circ G(Z_0) \simeq F(G(Z_0))$$

Recall: we saw that even if F & G are structure types, $F \circ G$ may not be a structure type (e.g. E^{E^Z}) – it <u>will</u> be a stuff type.

5) Given two stuff types F & G, define the "inner product" $\langle F, G \rangle$, a groupoid with

$$|\langle F, G \rangle| = \langle |F|, |G| \rangle$$

where the right-hand side is the inner product on the Hilbert space (Fock representation) for the quantum harmonic oscillator: given $f, g \in \mathbb{C}[z]$, this had

$$f(z) = \sum_{n \in \mathbb{N}} a_n z^n \qquad \Longrightarrow \quad \langle f, g \rangle = \sum_{n \in \mathbb{N}} n! \, \bar{a}_n b_n$$
$$g(z) = \sum_{n \in \mathbb{N}} b_n z^n$$

We saw this in the homework on "categorified coherent states," where we showed that $\langle 1, 1 \rangle = 1$ & $\langle a^* f, g \rangle = \langle f, a g \rangle$ determine the inner product

$$\langle z^n, z^m \rangle = n! \, \delta_{n,m}$$

The idea:

$$\begin{aligned} \langle z^n, z^n \rangle &= \langle a^* z^{n-1}, z^n \rangle \\ &= \langle z^{n-1}, a z^n \rangle \\ &= \langle z^{n-1}, \tfrac{d}{dz} z^n \rangle \\ &= n \langle z^{n-1}, z^{n-1} \rangle = \cdots = n! \langle 1, 1 \rangle = n! \end{aligned}$$

It's a minor miracle that this inner product comes naturally from category theory.

6) Define "stuff operators," which map stuff types to stuff types. E.g.:

A = "annihilation operator" with $|AF| = a|F|$

A^* = "creation operator" with $|A^*F| = a^*|F|$

7) Categorify the theory of Feynman diagrams.

The Generating Function of a Stuff Type

Suppose
$$\begin{array}{c} X \\ \downarrow F \\ \mathrm{FinSet}_0 \end{array}$$
is a stuff type. Define $|F| \in \mathbb{C}[[z]]$ by

$$|F|(z) = \sum_{n \in \mathbb{N}} |X_n| \, z^n$$

where $|X_n|$ is the groupoid cardinality (if it converges!) of X_n, the "groupoid of F-stuffed n-elt. sets," i.e. n-elt. sets equipped with F-stuff.

More precisely: note

$$\mathrm{FinSet}_0 \cong \sum_{n \in \mathbb{N}} [n\text{-elt sets}]_0$$

— a sum (aka. disjoint union or coproduct) of groupoids whose objects are n-elt. sets and whose morphisms are bijections. Therefore

$$X \cong \sum_{n \in \mathbb{N}} X_n$$

where the objects of X_n are those $x \in X$ for which $F(x)$ is an n-elt. set, and the morphisms are all morphisms in X between these.

So if X = the groupoid of "F-stuffed finite sets"
then X_n = the groupoid of "F-stuffed n-elt. sets,"
& we get

$$\begin{tikzcd} X_n \arrow[d, "F_n"'] \\ \text{[n-elt sets]}_0 \end{tikzcd}$$

where $F_n = F|_{X_n}$.

Note: unlike our old definition for structure types, there's no "$\frac{1}{n!}$" in this definition of the generating function — but it'll give the same result when F is a structure type!

Examples:

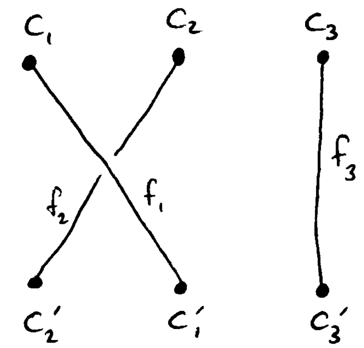
1) Categorified Coherent States, revisited.

Let F = "being a C-colored finite set," where C is any groupoid. If C is just the set k (viewed as a groupoid), this is "being a k-colored finite set" & we've seen that its generating function is $e^{kz} = \sum \frac{k^n}{n!} z^n$ since there are k^n ways to k-color an n-elt set. This is a structure type, but we've also mentioned "being a $\frac{1}{2}$-colored finite set" —where C is $\mathbb{Z}/2$ thought of as a groupoid with one object. This will be a stuff type.

More precisely, we have:

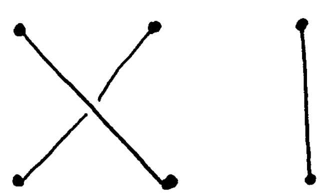
$$\begin{array}{l} X = \text{groupoid of } C\text{-colored finite sets} \\ \Big\downarrow F \quad \text{— forget the coloring} \\ \mathrm{FinSet}_0 \end{array}$$

Here X has objects finite sets with elements labelled by objects of C, & morphisms bijections with "strands" labelled by morphisms of C:



$f_i : c_i \to c_{\sigma(i)}$ where σ is a bijection

Applying F to this morphism we get



F forgets stuff because in general it's not faithful —2 different morphisms in X can have the same underlying bijection. If C has only identity morphisms (i.e. it's secretly a set), then F will be faithful, hence a structure type.

What's the generating function $|F|$?

We have:

$$\begin{array}{ccc} X & \simeq E^C = \sum_{n \in \mathbb{N}} \frac{C^n}{n!} \\ {\scriptstyle F}\big\downarrow & \\ \mathrm{FinSet}_0 & \simeq E = \sum_{n \in \mathbb{N}} \frac{1}{n!} \end{array}$$

& thus

$$\begin{array}{ccccc} X_n & = & \frac{C^n}{n!} & = & [C\text{-colored } n\text{-elt sets}]_0 \\ {\scriptstyle F_n}\big\downarrow & & & & \\ [n\text{-elt sets}]_0 & = & \frac{1}{n!} & & \end{array}$$

So

$$\begin{aligned} |F|(z) &= \sum_{n \in \mathbb{N}} |X_n| z^n \\ &= \sum_{n \in \mathbb{N}} \left| \frac{C^n}{n!} \right| z^n \\ &= \sum_{n \in \mathbb{N}} \frac{|C|^n}{n!} z^n = e^{|C| z} \end{aligned}$$

This is a coherent state of the harmonic oscillator with:

expected position $|C|/\sqrt{2}$

expected momentum 0

We can't yet get coherent states with nonzero expected momentum, or negative expected position, since we don't know gadgets with complex or negative cardinality.