13 April 2004

Stuff Types & their Generating Functions

Before doing more examples, let's review:

A stuff type is a functor

$$\begin{tikzcd} X \arrow[d, "F"'] \\ FinSet_0 \end{tikzcd}$$

where X is a groupoid which we call the groupoid of "F-stuffed sets." The generating function of this stuff type is:

$$|F|(z) = \sum_{n \in \mathbb{N}} |X_n| z^n$$

where X_n is the groupoid of "F-stuffed n-element sets" so that:

$$X = \sum_{n \in \mathbb{N}} X_n$$

$$FinSet_0 = \sum_{n \in \mathbb{N}} [n\text{-elt sets}]_0$$

& we have

$$\begin{tikzcd} X_n \arrow[d, "F_n"'] \\ {[n\text{-elt sets}]_0} \end{tikzcd} \qquad F_n = F|_{X_n}$$

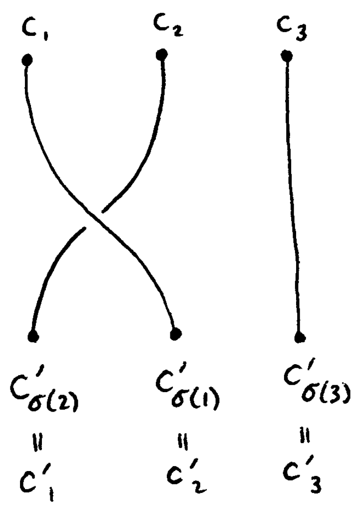
Our example so far was

F = "being a C-colored finite set"

where C is a groupoid. Here

$$\begin{array}{ccl} X_n & = & \text{the groupoid of "C-colored n-elt sets"} \\ {\scriptstyle F_n}\big\downarrow & & \text{— forgets the C-coloring} \\ [n\text{-elt sets}]_0 & & \end{array}$$

A morphism in X_n looks like:



$c_i \in C$

$\sigma : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$ (or any 3-elt set)

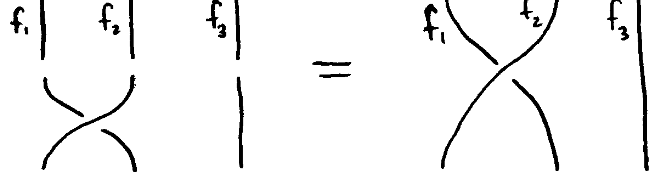
$f_i : c_i \longrightarrow c'_{\sigma(i)}$ are morphisms in C

So

$$X_n \simeq \frac{C^n}{n!}$$

$n!$ has an action on C^n by "crossing the strands"

objects are n-tuples of objects in C; morphisms in X_n are n-tuples of morphisms in C composed with permutations



So :

$$|F|(z) = \sum |X_n| z^n$$
$$= \sum \left| \frac{C^n}{n!} \right| z^n$$
$$= \sum \frac{|C|^n}{n!} z^n$$
$$= e^{|C|z}$$

Q: What should this stuff type really be called?

A: "E^{CZ}" so that $|E^{CZ}| = e^{|C|z}$.

Q: What does this really <u>mean</u>? I.e. if E^{CZ} is something like a "function", what sort of values does the argument Z take?

A: Groupoids! If Z_0 is a groupoid, E^{CZ_0} should be a groupoid — the groupoid of "CZ_0-colored finite sets", i.e. finite sets with elts labelled by objects of CZ_0, i.e. pairs (c, z) with $c \in C$, $z \in Z_0$.

Last quarter we saw that $|E^{CZ_0}| = e^{|C||Z_0|}$.

E.g.: $C = \frac{1}{2!}$ ($\mathbb{Z}/2$ seen as a groupoid)

Then

$$E^{CZ} = \text{"being a } \tfrac{1}{2}\text{-colored finite set"}$$

and if we evaluate this at $Z_0 = 1$ (the 1-elt set, viewed as a groupoid)

$$\circlearrowleft 1$$

we get

$$E^C = E^{1/\!/2!} = \text{the groupoid of } \tfrac{1}{2}\text{-colored finite sets}$$
$$\simeq \text{Cubes}$$

(see p. 67, Week 7, Winter 2004)

Back to our examples:

2) F = "being the first of two finite sets of the same cardinality"

We want to make this a stuff type

$$\begin{tikzcd} X \arrow[d, "F"'] \\ \mathrm{FinSet}_0 \end{tikzcd}$$

where objects of X are pairs of sets (S,T) with $S \cong T$, & morphisms are pairs of bijections, and F forgets the second set and second morphism in these pairs. This is a stuff type but not a structure type because it's not faithful.

$$X_n = \text{the groupoid of "pairs of } n\text{-elt sets"}$$
$$\simeq [n\text{-elt sets}]_0 \times [n\text{-elt sets}]_0$$

So

$$|X_n| = \left|[n\text{-elt sets}]_0\right|^2$$

&

$$[n\text{-elt sets}] \simeq \frac{1}{n!} \qquad \text{so} \qquad |X_n| = \frac{1}{(n!)^2}$$

and so $$|F|(z) = \sum_{n\in\mathbb{N}} \frac{z^n}{(n!)^2}$$ (which can be expressed using integrals of Bessel fns.)

Note: This couldn't possibly be the generating function of any structure type, since if we write

$$|F|(z) = \sum \frac{a_n}{n!} z^n$$

we don't get $a_n \in \mathbb{N}$! (Of course we are ignoring the detail that we haven't shown our way of finding generating functions of stuff types is equivalent to our old way for structure types — we'll deal with this later)

Now we can fill in this chart:

If F is a...	then $\lvert F\rvert(z) = \sum \frac{a_n}{n!} z^n$ where:	since these numbers are cardinalities of:
stuff type	$a_n \in \mathbb{R}^+ = [0,\infty)$	(tame) groupoids = 1-groupoids
structure type	$a_n \in \mathbb{N}$	(finite) sets = 0-groupoids
property type	$a_n \in \{0,1\} \cong \{F,T\}$	truth values = -1-groupoids
vacuous property type	$a_n \in \{1\} \cong \{T\}$	true = the only -2-groupoid

The reason this all works in this way is that in equipping a finite set with extra $\left\{\begin{array}{l}\text{stuff}\\ \text{structure}\\ \text{properties}\\ \text{vacuous properties}\end{array}\right.$, there's

a $\left\{\begin{array}{l}\text{groupoid}\\ \text{set}\\ \text{pair}\\ \text{1-elt set}\end{array}\right.$, i.e a $\left\{\begin{array}{l}\text{1-groupoid}\\ \text{0-groupoid}\\ \text{-1-groupoid}\\ \text{-2-groupoid}\end{array}\right.$ of choices.

15 April 2004

Evaluation & Composition of Stuff Types

Following what we did last quarter for structure types, let's define $F(Z_0)$, where F is a stuff type:

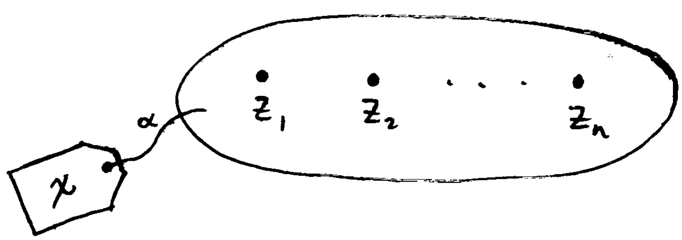
$$\begin{array}{c} X \quad \text{--- a groupoid} \\ {\scriptstyle F}\big\downarrow \\ \mathrm{FinSet}_0 \end{array}$$

& Z_0 is any groupoid. Copying what we did, let

$$F(Z_0) = \text{the groupoid of "}F\text{-stuffed } Z_0\text{-colored finite sets"}$$

just as before, but with "stuffed" instead of "structured": now we're equipping the finite sets with extra stuff.

A typical object in $F(Z_0)$ looks like:



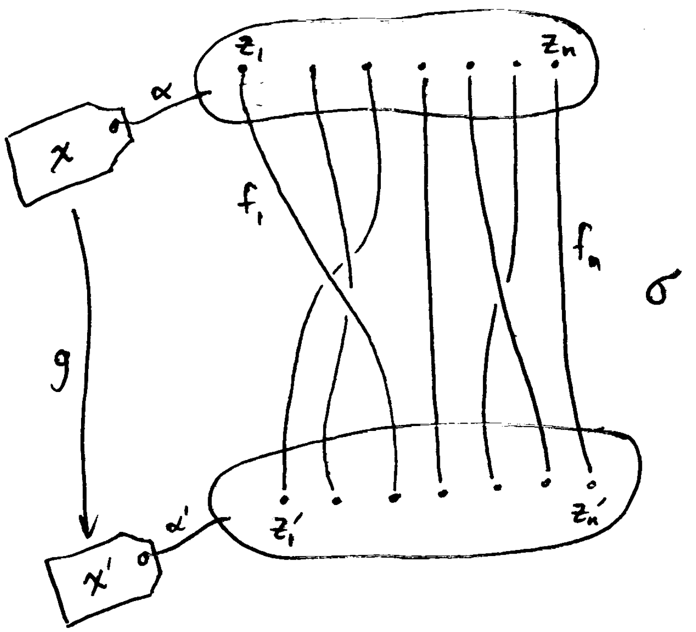
$z_1, \ldots, z_n \in Z_0$ label or "color" the elements of the finite set $\{a_1, \ldots, a_n\}$

$x \in X$ is an object equipped with an (iso)morphism $\alpha : F(x) \xrightarrow{\sim} \{a_1, \ldots, a_n\}$

redundant since FinSet_0 is a groupoid

The "tag" is our cute depiction of the "F-stuff"
α is the string that connects the "F-stuff" to the set.

A typical morphism in $F(Z_0)$ looks like:



$\sigma : \{a_1, \dots, a_n\} \xrightarrow{\sim} \{a_1', \dots, a_n'\}$

$f_i : z_i \longrightarrow z'_{\sigma(i)}$
morphisms in Z_0

$g : x \to x'$ a morphism in X

such that

$$\begin{tikzcd} F(x) \arrow[r, "\alpha"] \arrow[d, "F(g)"'] & \{a_1, \dots, a_n\} \arrow[d, "\sigma"] \\ F(x') \arrow[r, "\alpha'"] & \{a_1', \dots, a_n'\} \end{tikzcd}$$

commutes

<u>Thm:</u>

$$|F(Z_0)| = |F|(|Z_0|)$$

<u>Proof:</u> Haven't you noticed we never prove theorems here? ∎

Given two stuff types:

$$\begin{array}{ccc} X & \quad & Y \\ {\scriptstyle F}\big\downarrow & & {\scriptstyle G}\big\downarrow \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{array}$$

can we compose them to get a stuff type $F \circ G$ with

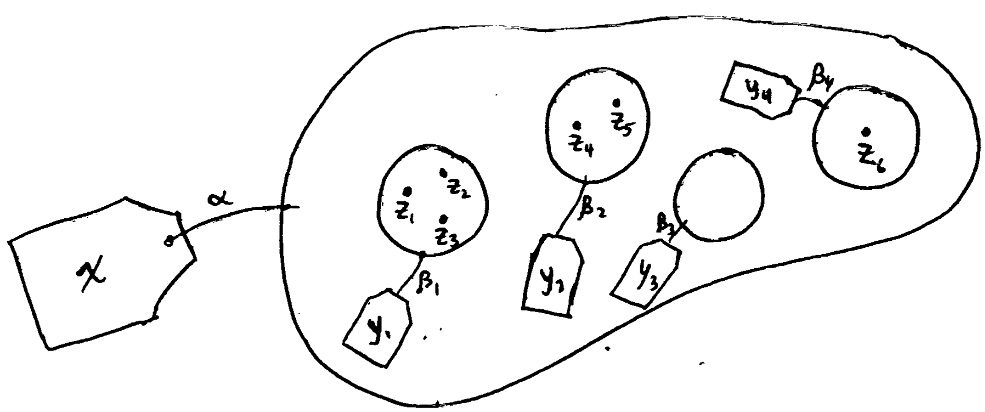
$$(F \circ G)(Z_0) \simeq F(G(Z_0)) \quad ?$$

Yes!

$F(G(Z_0))$ = the groupoid of "F-stuffed $G(Z_0)$-colored finite sets"

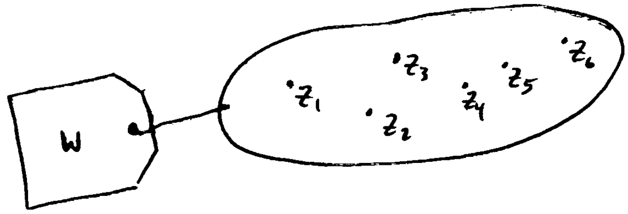
= the groupoid of "F-stuffed finite sets with elements labelled by objects of $G(Z_0)$"

= the groupoid of "F-stuffed finite sets with elements labelled by G-stuffed Z_0-colored finite sets."



This groupoid is the same as:

$(F \circ G)(Z_0) :=$ the groupoid of "$F \circ G$-stuffed Z_0-colored finite sets"



$$W \xrightarrow{F \circ G} \mathrm{FinSet}_0$$

$w \in W$

provided we let

$F \circ G$ = "being a finite set S written as a finite disjoint union $S_1 + \cdots + S_n$ with each S_i equipped with G-stuff and $\{1, \ldots, n\}$ equipped with F-stuff."

corr. to $S_1, \ldots, S_n$

= "being the disjoint union of a finite F-stuffed family of G-stuffed finite sets."

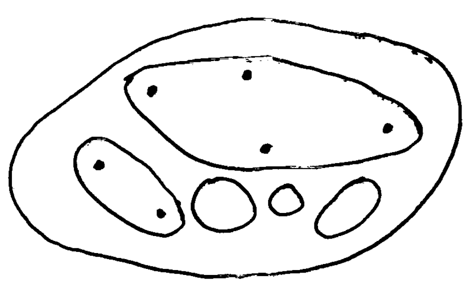
Examples:

1) $\mathrm{COSH}\ Z$ = "being an even set"

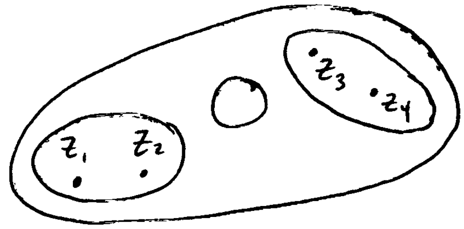
E^Z = "being a finite set"

$E^{\mathrm{COSH}\ Z}$ = "being a disjoint union of a finite family of even sets

E.g.



$E^{COSH(Z_0)}$ = the groupoid of "finite disjoint unions of Z_0-colored even sets"

typical object: 

in particular:

$E^{COSH(1)}$ = the groupoid of "finite disjoint unions of 1-colored even sets"

$\simeq$ the groupoid of "finite disjoint unions of even sets"

and

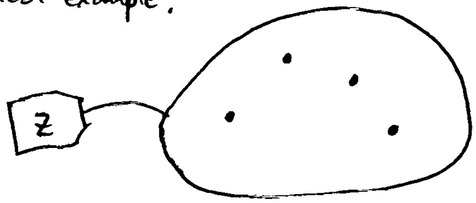
$$|E^{COSH(1)}| = e^{|COSH(1)|} = e^{\cosh 1} = e^{\frac{e+e^{-1}}{2}} .$$

What's $E^{Z_0 COSH(1)}$?

$COSH(1) \simeq$ the groupoid of "even sets"

$Z_0 COSH(1) \simeq$ the groupoid of pairs (z_0, x) w/ $z_0 \in Z_0$ & $x \in COSH\,1$

a typical example:



$z \in Z_0$

So $E^{Z_0 COSH(1)} \simeq$ the groupoid of "finite sets w. elts labelled by pairs $(z,x) \in Z_0 COSH(1)$":