20 April 2004

Structure Types – New vs. Old Definition

How are our new structure types:

$$\begin{tikzcd} X \arrow[d, "F"] & \text{groupoid} \\ \mathrm{FinSet}_0 & \end{tikzcd}$$

F ← faithful functor

secretly the same as our old ones:

$$\mathrm{FinSet}_0 \xrightarrow{G} \mathrm{Set} \qquad ?$$

G ← functor

We'll show how to get $G = F^*$ from F. In fact, we can even do something like this for arbitrary stuff types:

$$\begin{tikzcd} X \arrow[d, "F"] & \text{groupoid} \\ \mathrm{FinSet}_0 & \end{tikzcd}$$

F ← functor

We want to define for any $S \in \mathrm{FinSet}_0$ its fiber (i.e. its "inverse image" $F^{-1}(S)$) consisting of the <u>set</u> of all ways to put F-<u>structure</u> on S, or more generally the <u>groupoid</u> of ways to put F-stuff on S. If we do this, we'll get

$$\begin{aligned} G \doteq F^* : \mathrm{FinSet}_0 &\longrightarrow \mathrm{Gpd} \\ S &\longmapsto F^{-1}(S) \end{aligned}$$

the 2-category of groupoids.

but when F is faithful this will reduce to:

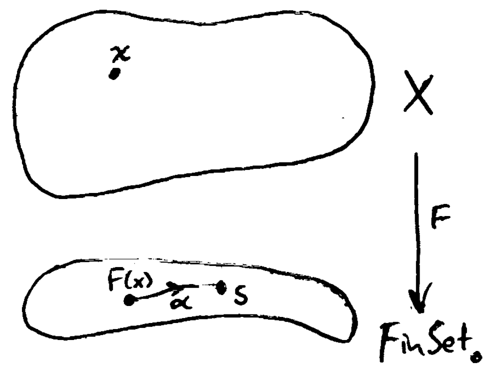
$$G = F^* : \mathrm{FinSet}_0 \longrightarrow \mathrm{Set} \subseteq \mathrm{Gpd}$$

—our original description of structure types.

So: given $\begin{array}{c} X \\ \downarrow F \\ \mathrm{FinSet} \end{array}$ & given $S \in \mathrm{FinSet}_0$ let's

define $F^{-1}(S)$ to be the groupoid s.t.:

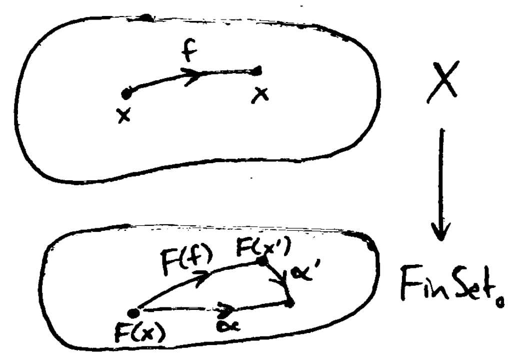
- an <u>object</u> of $F^{-1}(S)$ is an object $x \in X$ equipped with an isomorphism
  $$\alpha : F(x) \xrightarrow{\sim} S$$



- a <u>morphism</u> in $F^{-1}(S)$ from (x,α) to (x',α') is a morphism $f : x \to x'$ s.t.

$$\begin{tikzcd} F(x) \arrow[rr, "F(f)"] \arrow[dr] & & F(x') \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.



Technically we call $F^{-1}(S)$ the <u>weak inverse image</u> of S under F, or the <u>homotopy fiber</u>, following terminology from topology.

We can extend this process to get a weak 2-functor:

$$F^*: \mathrm{FinSet}_0 \longrightarrow \mathrm{Gpd}$$
$$S \longmapsto F^{-1}(S) = F^*(S)$$

Note: thinking of FinSet_0 as a 2-Cat with only identity 2-morphisms.

i.e. for any bijection $f: S \to S'$ we get

$$F^*(f): F^*(S) \longrightarrow F^*(S')$$

s.t. $F^*(ff')$ equals $F^*(f)F^*(f')$ only up to specified natural isomorphism, which satisfy some coherence laws.

If F is a structure type, i.e. it's faithful, then $F^{-1}(S)$ is (a groupoid that's equivalent to) a set. Then by tweaking F^* slightly to make $F^{-1}(S)$ really be a set, we get

$$F^*: \mathrm{FinSet}_0 \longrightarrow \mathrm{Set} \subseteq \mathrm{Gpd}$$

— our old way of thinking about structure types.

Conversely, any "old" structure type gives us a "new" one.. So...

A functor $F: X \to \mathrm{FinSet}_0$ that forgets ______ ...	is the same as
stuff	$F^*: \mathrm{FinSet}_0 \to \mathrm{Gpd} = 1\text{-Gpd}$
structure	$F^*: \mathrm{FinSet}_0 \to \mathrm{Set} = 0\text{-Gpd}$
properties	$F^*: \mathrm{FinSet}_0 \to \{\emptyset, 1\} = -1\text{-Gpd}$
vacuous properties	$F^*: \mathrm{FinSet}_0 \to \{1\} = -2\text{-Gpd}$

In short, there's a

$$\left\{\begin{array}{l}\text{groupoid}\\ \text{set}\\ \text{empty or 1-elt set}\\ \text{1-elt set}\end{array}\right. \text{ of ways to equip a finite set w. } \left\{\begin{array}{l}\text{stuff}\\ \text{structure}\\ \text{property}\\ \text{vacuous property}\end{array}\right.$$

of some type.

The Inner Product of Stuff Types

First consider

$$\ell^2 = \{\psi : \mathbb{N} \longrightarrow \mathbb{C} \; : \; \sum_{n \in \mathbb{N}} |\psi_n|^2 < \infty\}.$$

This is a Hilbert space with inner product

$$\langle \varphi, \psi \rangle := \sum_{n \in \mathbb{N}} \overline{\varphi}_n \psi_n$$

Following our general strategy, we can replace $\mathbb{C}$ by its combinatorial heart, $\mathbb{N}$, getting:

$$\{\psi : \mathbb{N} \longrightarrow \mathbb{N} \; : \; \sum_{n \in \mathbb{N}} \psi_n^2 < \infty\}$$

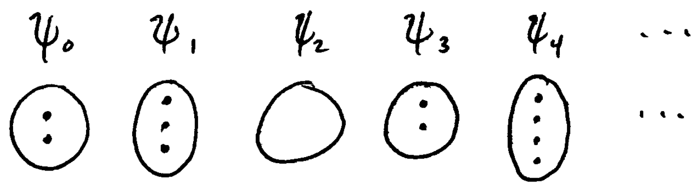
with inner product

$$\langle \varphi, \psi \rangle := \sum_{n \in \mathbb{N}} \varphi_n \psi_n \in \mathbb{N}$$

Then we could categorify this a bit, getting:

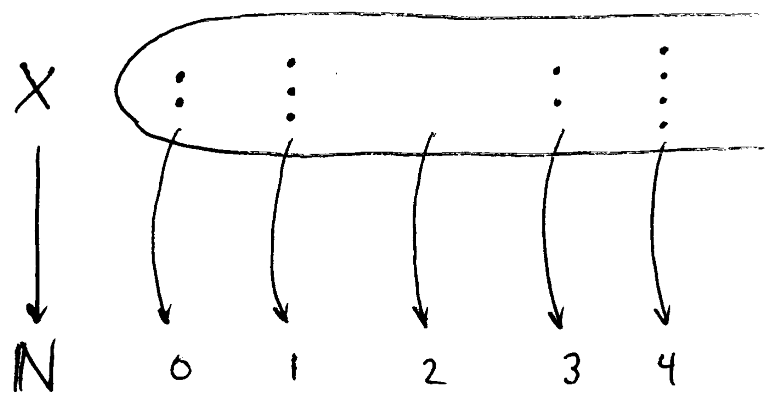
$$\left\{ \psi : \mathbb{N} \longrightarrow \mathrm{FinSet} \;\; : \;\; \left| \sum_{n \in \mathbb{N}} \psi_n^2 \right| < \infty \right\}$$

Such a ψ looks like:



only finitely many nonempty

or:



where X is a finite set.

In fact $\psi = F^*$ is defined in terms of F just as we'd seen before:

$$\psi_n = F^{-1}(n)$$

— i.e. in terms of inverse images.

$$\underset{\substack{\cap \\ \mathrm{FinSet}}}{F : X} \longrightarrow \mathbb{N} \quad \text{is "the same" as} \quad \overset{F^*}{\overset{\parallel}{\psi}} : \mathbb{N} \longrightarrow \mathrm{FinSet}$$

$$(\text{s.t } |\Sigma \psi_n^2| < \infty)$$

This is just like:

$$\underset{\underset{\text{Gpd}}{\Uparrow}}{F : X} \longrightarrow \text{FinSet}_0 \quad \text{is "the same" as} \quad \overset{\overset{F^*}{\|}}{\psi} : \text{FinSet}_0 \longrightarrow \text{Gpd}$$

(s.t. something is finite...?)

We can use this analogy to define the inner product for stuff types.

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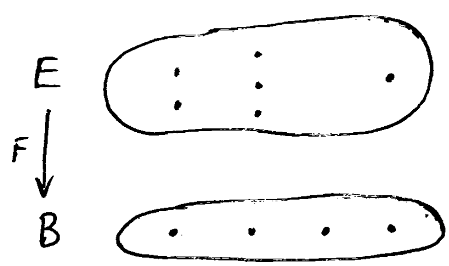
The Inner Product of Stuff Types (cont.)

What we've seen is that any function $F : E \to B$ can be seen as a "bundle" with "total space" E & "base space" B:

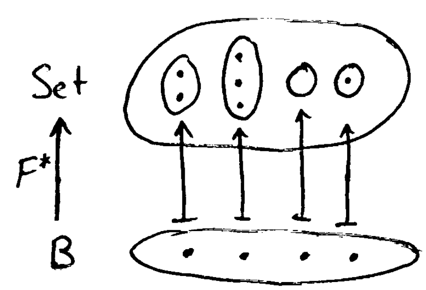


with F the "projection". We can also think of it as a functor

$$F^* : B \longrightarrow \text{Set}$$

(viewing B as a category w/ only identity morphisms) assigning to each $b \in B$ the "fiber" (inverse image)

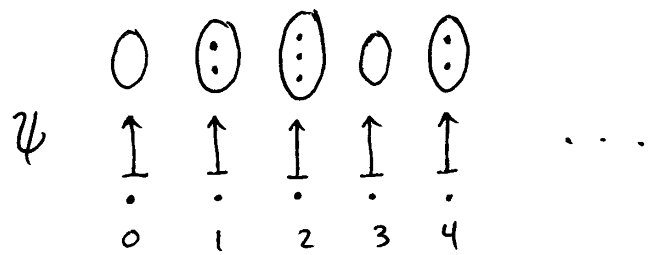
$$F^*(b) = F^{-1}(b) \quad :$$



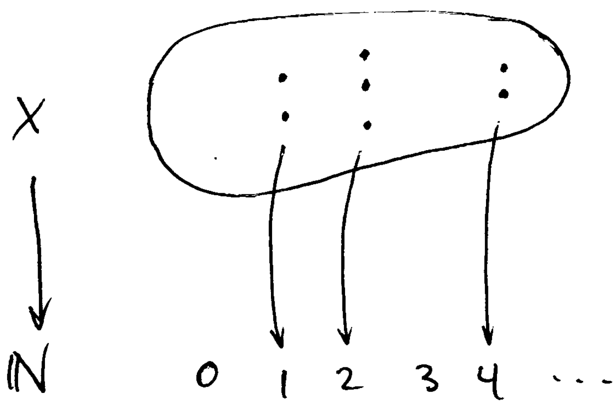
The second way of thinking about functions seems more complicated, but that's the one that comes up when you categorify ℓ^2 a little bit, getting:

$$\mathcal{L}^2 = \{ \psi : \mathbb{N} \to \mathrm{Set} \; : \; \sum |\psi_n|^2 < \infty \}$$

$|\cdot| :=$ cardinality



Here ψ_n is the "set of ways for possibility #n to occur." (Contrast this to $\psi_n \in \mathbb{C}$ in ordinary QM — the "amplitude for possibility #n to occur.") But let's think about this "categorified ℓ^2" the first way: $\psi = F^*$ for some bundle $F : X \to \mathbb{N}$



and the condition that ψ be normalizable: $\sum |\psi_n|^2 < \infty$ becomes just the condition that X is a finite set.

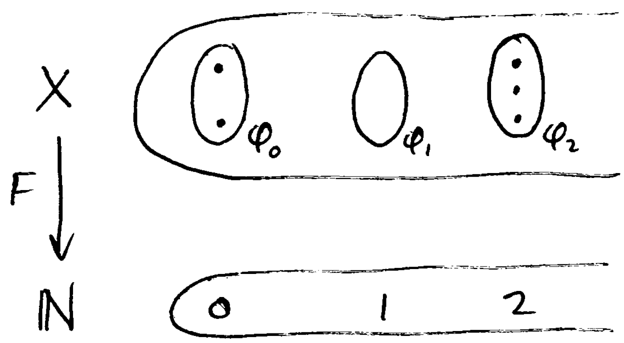
Suppose we have two elements $\varphi, \psi \in \mathcal{L}^2$ corresponding to two "bundles" $F : X \longrightarrow \mathbb{N}$, $G : Y \longrightarrow \mathbb{N}$ $(X, Y \in \mathrm{FinSet})$:

$$\varphi = F^* \qquad \psi = G^*$$

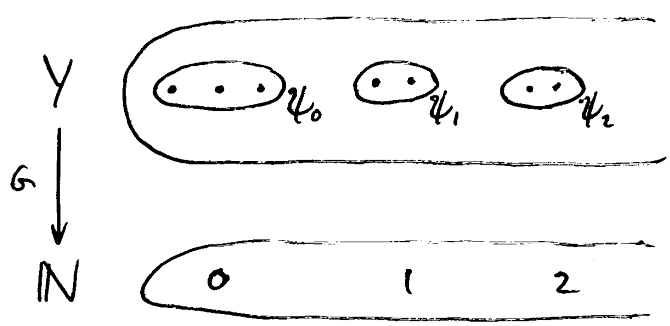
On the other hand, we can define their "inner product" to be:

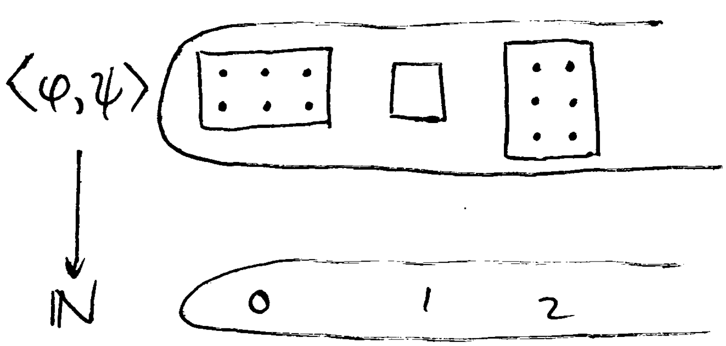
$$\langle \varphi, \psi \rangle = \sum_{n \in \mathbb{N}} \varphi_n \psi_n \in \mathrm{FinSet}$$

but how can we describe this in terms of F & G?



$\varphi_n := F^{-1}(n)$





This is the "<u>fiberwise product</u>" of the bundles $F : X \to \mathbb{N}$, $G : Y \to \mathbb{N}$. The set $\langle \varphi, \psi \rangle$ is called $X \times_{\mathbb{N}} Y$.

We have

$$X \times_{\mathbb{N}} Y = \sum_{n \in \mathbb{N}} \varphi_n \psi_n = \sum_{n \in \mathbb{N}} F^{-1}(n) \times G^{-1}(n)$$

$$= \{ (x,y) \in X \times Y \ : \ F(x) = G(y) \}$$

this is the way people usually write down the definition of fiberwise product.

and maps making the following diagram commute.

$$\begin{tikzcd}
& X \times_{\mathbb{N}} Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\
X \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\
& \mathbb{N} &
\end{tikzcd}$$

$p_X(x,y) = x$
$p_Y(x,y) = y$

In fact, $X \times_{\mathbb{N}} Y$ is initial among such gadgets - it's called the <u>pullback</u> of

$$\begin{tikzcd}
X \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\
& \mathbb{N} &
\end{tikzcd}$$

Let's recall the definition of pullback. In any category a pullback of the diagram

$$\begin{tikzcd} x \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & z & \end{tikzcd}$$

is an object a with morphisms $p_x : a \to x$, $p_y : a \to y$ s.t.

$$\begin{tikzcd} & a \arrow[dl, "p_x"'] \arrow[dr, "p_y"] & \\ x \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & z & \end{tikzcd}$$

commutes & a is initial among such: if a', $p_x' : a' \to x$, $p_y' : a' \to y$ is another one (making the analogous square commute) then $\exists!\ f : a \to a'$ s.t.

$$\begin{tikzcd} & a \arrow[d] \arrow[ddl, bend right, "p_x"'] \arrow[ddr, bend left, "p_y"] & \\ & a' \arrow[dl, "p_x'"'] \arrow[dr, "p_y'"] & \\ x \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & z & \end{tikzcd}$$

commutes.

In the category Set, we can take $a = \{(x_0, y_0) \in x \times y : f(x_0) = g(y_0)\}$ — i.e. a fiberwise product!

Now that we know how to take the "inner product" of

$$F: \underset{\text{Set}}{\underset{\cap}{X}} \longrightarrow \mathbb{N} \quad \& \quad G: \underset{\text{Set}}{\underset{\cap}{Y}} \longrightarrow \mathbb{N}$$

we can copy this and get the inner product of stuff types:

$$F: \underset{\text{Gpd}}{\underset{\cap}{X}} \longrightarrow \text{FinSet}_0 \quad \& \quad G: \underset{\text{Gpd}}{\underset{\cap}{Y}} \longrightarrow \text{FinSet}_0$$

The inner product of stuff types F & G will be a groupoid $\langle F, G \rangle$. This will be defined as a <u>weak pullback</u>:

$$\langle F, G \rangle = X \times_{\text{FinSet}_0} Y$$

$$\begin{tikzcd}
 & X \times_{\text{FinSet}_0} Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\
X \arrow[rr, phantom, "\overset{\alpha}{\underset{\sim}{\Longrightarrow}}"] \arrow[dr] & & Y \arrow[dl] \\
 & \text{FinSet}_0 &
\end{tikzcd}$$

Now this will only commute up to the natural isomorphism α, and only be <u>weakly</u> initial among such gadgets. It should come as no surprise that $X \times_{\text{FinSet}_0} Y$ can be defined as the groupoid where:

— an object is a pair $(x,y) \in X \times Y$ equipped with an isomorphism $\alpha_{(x,y)} =: \alpha : F(x) \xrightarrow{\sim} G(y)$.

— a morphism is a morphism $(f,g) : (x,y) \longrightarrow (x',y')$ in $X \times Y$ such that:

$$\begin{tikzcd} F(x) \arrow[r, "\alpha"] \arrow[d, "F(f)"'] & G(y) \arrow[d, "G(g)"] \\ F(x') \arrow[r, "\alpha'"] & G(y') \end{tikzcd}$$

commutes.