27 April 2004

## The Inner Product of Stuff-Types – the key example

Recall that given two stuff types

$$F : X \longrightarrow \mathrm{FinSet}_0 \qquad G : Y \longrightarrow \mathrm{FinSet}_0$$

their inner product is a groupoid $\langle F, G \rangle$ defined as the weak pull back

$$\begin{tikzcd} & \langle F,G \rangle \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] & \\ X \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\ & \mathrm{FinSet}_0 & \end{tikzcd}$$

this notation just means the square is a (weak) pullback

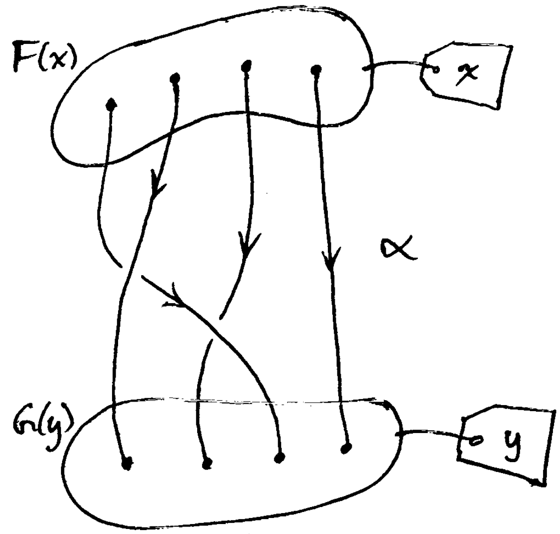
Recall:

- An object of $\langle F, G \rangle$ is an object $x \in X$, an object $y \in Y$, and an isomorphism $\alpha : F(x) \xrightarrow{\sim} G(y)$.
- A morphism in $\langle F, G \rangle$ is a morphism $f : x \to x'$ in $X$, a morphism $g : y \to y'$ in $Y$ s.t.

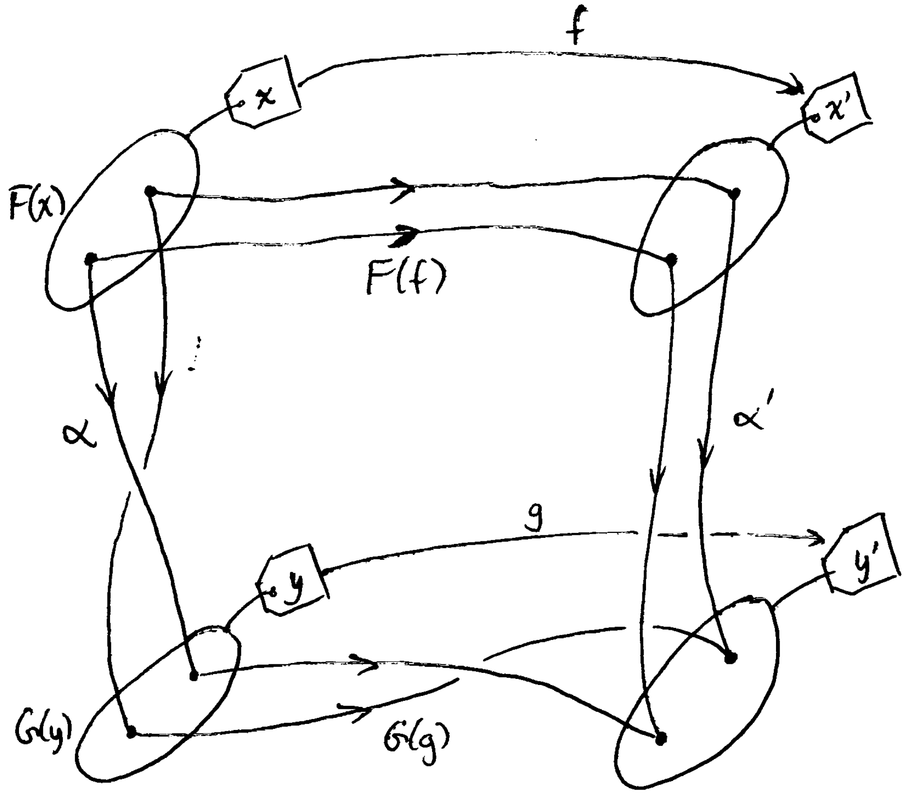
$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha"'] & F(x') \arrow[d, "\alpha'"] \\ G(y) \arrow[r, "G(g)"] & G(y') \end{tikzcd}$$

commutes.

An object in $\langle F, G \rangle$ looks like




A morphism looks like




Note: $\alpha\, G(g) = F(f)\, \alpha'$.

The square commutes.

## The key example:

We know the inner product on Fock space has

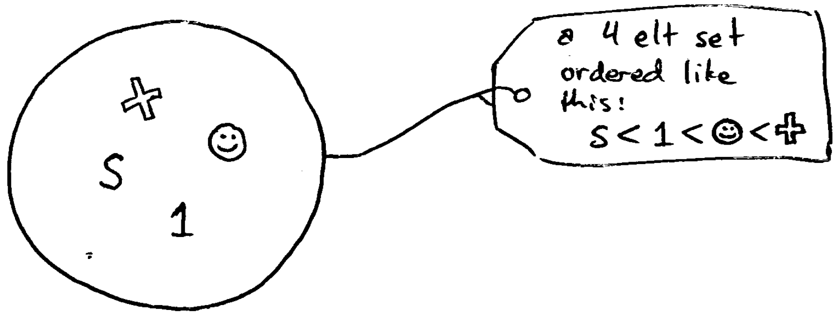
$$\langle z^n, z^m \rangle = n!\, \delta_{nm}$$

So what about

$$\langle Z^n, Z^m \rangle \;?$$

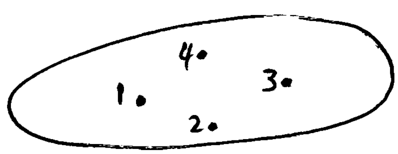
Recall: $Z^n$ = "being a totally ordered $n$-elt. set"

so a $Z^n$-stuffed set looks like:



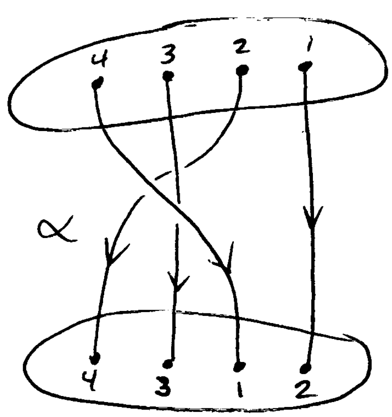

(if $n=4$)

or, with our usual habit of drawing sets so all elements look the same:




(Here we indicate the $Z^4$-stuff — namely the ordering and the property of having 4 elts. — not by a 'tag' but simply by enumerating the elts)

and an object of $\langle Z^n, Z^m \rangle$ looks like this




Note: $\alpha$ is just any bijection.

If $n \neq m$, there are no bijections, so no objects in $\langle Z^n, Z^m \rangle$. So:

$$\langle Z^n, Z^m \rangle = \emptyset \qquad \text{if } n \neq m$$

the groupoid with *no* objects

If $n = m$, there are $n!$ such bijections, & with some thought we get

$$\langle Z^n, Z^n \rangle \simeq n!$$

where $n!$ is the groupoid with $n!$ objects and only identity morphisms.

So we see:

$$\langle Z^n, Z^m \rangle \simeq n!\, \delta_{nm}$$

where

$$\delta_{nm} = \begin{cases} \text{the empty groupoid, } 0, & \text{if } n \neq m \\ \text{the one-object groupoid, } 1, & \text{if } n = m \end{cases}$$

& so

$$|\langle Z^n, Z^m \rangle| = \langle |Z^n|, |Z^m| \rangle$$

where the right hand side is inner product in Fock space.

# Stuff Operators

A stuff type is:

$$
\begin{tikzcd}
X \arrow[d, "F"'] & \in \mathrm{Gpd} \\
\mathrm{FinSet}_0
\end{tikzcd}
$$

The inner product of two stuff types is:

$$
\begin{tikzcd}
& \langle F, G \rangle \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& \mathrm{FinSet}_0 &
\end{tikzcd}
$$

A stuff operator is:

$$
\begin{tikzcd}
& T \arrow[dl, "P"'] \arrow[dr, "Q"] & \in \mathrm{Gpd} \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}
$$

To apply a stuff operator to a stuff type we do a pullback:

$$
\begin{tikzcd}
& & TF \arrow[dl, "p_T"'] \arrow[dr, "p_X"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right] & \\
& T \arrow[dl, "P"'] \arrow[dr, "Q"] & & X \arrow[dl, "F"] \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0 &
\end{tikzcd}
$$

to get this stuff type:

$$
\begin{tikzcd}
TF \arrow[d, "p_T P"] \\
\mathrm{FinSet}_0
\end{tikzcd}
$$

To compose two stuff operators $S$ & $T$ we do:

$$\begin{tikzcd}[column sep=small]
 & & ST \arrow[dl, "P_S"'] \arrow[dr, "P_T"] \arrow[d, phantom, "\vee"] & & \\
 & S \arrow[dl, "M"'] \arrow[dr, "N"] & {} & T \arrow[dl, "P"'] \arrow[dr, "Q"] & \\
FinSet_0 & & FinSet_0 & & FinSet_0
\end{tikzcd}$$

to get the stuff operator:

$$\begin{tikzcd}
 & ST \arrow[dl, "P_S M"'] \arrow[dr, "P_T Q"] & \\
FinSet_0 & & FinSet_0
\end{tikzcd}$$

CATEGORIFICATION $\Longrightarrow$

| $\ell^2$ & operators on it | $\mathcal{L}^2$ & operators on it | Stuff types & stuff operators |
|---|---|---|
| $\ell^2$ consists of (certain) sequences<br>$\psi : \mathbb{N} \to \mathbb{C}$<br>$i \mapsto \psi_i \in \mathbb{C}$ | $\mathcal{L}^2$ consists of (certain) sequences<br>$\psi : \mathbb{N} \to Set$<br>$i \mapsto \psi_i \in Set$<br>which are the same as:<br>$X \in Set \xrightarrow{F} \mathbb{N}$<br>where $\psi_i = F^{-1}(i)$<br>– inverse image of $i \in \mathbb{N}$ | Stuff types are<br>$\psi : FinSet_0 \to Gpd$<br>$i \mapsto \psi_i \in Gpd$<br>which are the same as:<br>$X \in Gpd \xrightarrow{F} FinSet_0$<br>where $\psi_i = F^{-1}(i)$<br>– <u>weak</u> inverse image of $i \in FinSet_0$ |

Operators on $\ell^2$ are (certain) matrices

$$T : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$$

$$T_{ij} \in \mathbb{C}$$

Operators on $\mathcal{L}^2$ are certain matrices

$$T : \mathbb{N} \times \mathbb{N} \longrightarrow \mathrm{Set}$$

$$T_{ij} \in \mathrm{Set}$$

which are the same as

$$\begin{tikzcd} & X \in \mathrm{Set} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{N} & & \mathbb{N} \end{tikzcd}$$

where $T_{ij} = (p_1 \times p_2)^{-1}(i,j)$

– inverse image of $(i,j) \in \mathbb{N}^2$

A stuff operator is

$$T : \mathrm{FinSet}_0 \times \mathrm{FinSet}_0 \longrightarrow \mathrm{Gpd}$$

$$T_{ij} \in \mathrm{Gpd}$$

which are the same as

$$\begin{tikzcd} & X \in \mathrm{Gpd} \arrow[dl] \arrow[dr] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

where $T_{ij} = (p_1 \times p_2)^{-1}(i,j)$

– weak inv. image of $(i,j) \in \mathrm{FinSet}_0^{\,2}$

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An operator (or matrix)

$$T : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}$$

acts on a vector

$$\psi : \mathbb{N} \longrightarrow \mathbb{C}$$

to give a vector

$$T\psi : \mathbb{N} \longrightarrow \mathbb{C}$$

by:

$$(T\psi)_i = \sum_{j \in \mathbb{N}} T_{ij} \psi_j$$

if the sum converges

An "operator" (or matrix of sets)

$$T : \mathbb{N} \times \mathbb{N} \longrightarrow \mathrm{Set}$$

acts on a "vector"

$$\psi : \mathbb{N} \longrightarrow \mathrm{Set}$$

to give a vector

$$T\psi : \mathbb{N} \longrightarrow \mathrm{Set}$$

by

$$(T\psi)_i = \sum_{j \in \mathbb{N}} T_{ij} \times \psi_j$$

A stuff operator

$$T : \mathrm{FinSet}_0 \times \mathrm{FinSet}_0 \longrightarrow \mathrm{Gpd}$$

acts on a stuff type

$$\psi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Gpd}$$

to give a stuff type

$$T\psi : \mathrm{FinSet}_0 \longrightarrow \mathrm{Gpd}$$

which is best described using a weak pullback ...

We can also think of $T$ this way:

$$\begin{tikzcd} & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{N} & & \mathbb{N} \end{tikzcd}$$

where $T_{ij} = (p_1 \times p_2)^{-1}(i,j)$

and $\psi$ this way:

$$\begin{tikzcd} Y \arrow[d, "F"'] \\ \mathbb{N} \end{tikzcd}$$

$\psi_j = F^{-1}(j)$

and then $T\psi$ is defined using a pullback:

$$\begin{tikzcd} & & Z \arrow[dl] \arrow[dr] \arrow[dll, bend right, "G"'] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & Y \arrow[dl, "F"] \\ \mathbb{N} & & \mathbb{N} & \end{tikzcd}$$

getting

$$\begin{tikzcd} Z \arrow[d, "G"'] \\ \mathbb{N} \end{tikzcd}$$

& define $T\psi$ by

$$(T\psi)_i = G^{-1}(i)$$

Note: an elt. of $Z$ is a pair $(x,y) \in X \times Y$ s.t. $p_2(x) = F(y)$, i.e. an elt. of $T_{ij} \times \psi_j$ for some $i,j$, i.e. an elt. of

$$\sum_j T_{ij} \times \psi_j$$

for some $i$

$$\begin{tikzcd} & X \in Gpd \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ FinSet_0 & & FinSet_0 \end{tikzcd}$$

where $T_{ij} = (p_1 \times p_2)^{-1}(i,j)$ <u>weak</u> inverse image

$$\begin{tikzcd} Y \in Gpd \arrow[d] \\ FinSet_0 \end{tikzcd}$$

$\psi_j = F^{-1}(j)$ <u>weak</u> inv. image

and the $T\psi$ is defined using a <u>weak</u> pullback:

$$\begin{tikzcd} & & Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[rr, phantom, "\Rightarrow \alpha"] & & Y \arrow[dl, "F"] \\ FinSet_0 & & FinSet_0 & \end{tikzcd}$$

getting

$$\begin{tikzcd} Z \arrow[d, "G"'] \\ FinSet_0 \end{tikzcd}$$

$S_0$:

$$G^{-1}(i) = \sum_j T_{ij} \times \Psi_j$$

as desired.

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For the purposes of quantum mechanics (and QFT) it's nice to have some slightly different ways of drawing stuff types and stuff operators. Since stuff types are like <u>states</u> (vectors in Fock space), let's denote them like this:

$$\begin{tikzcd}
\Psi \in \mathrm{Gpd} \arrow[d, "p"'] \\
\mathrm{FinSet}_0
\end{tikzcd}$$

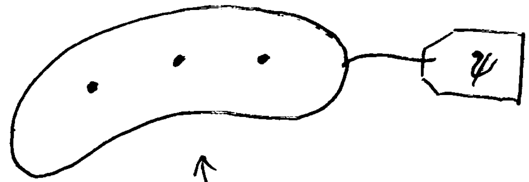
Since stuff operators are like operators on Fock space, let's denote them like this:

$$\begin{tikzcd}
& T \in \mathrm{Gpd} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}$$

When we act on a stuff type by a stuff operator, we'll write it like this:

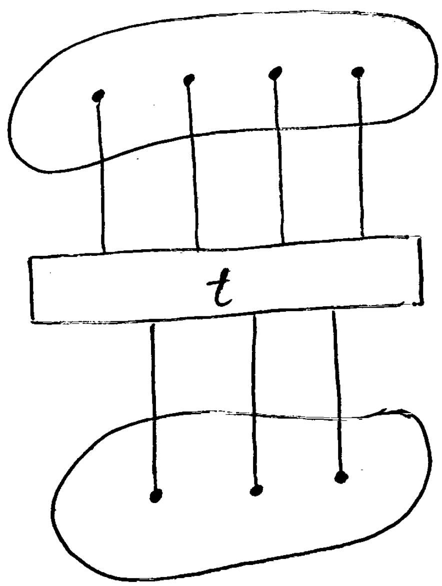
$$\begin{tikzcd}
& T\Psi \arrow[dl] \arrow[dr] & \\
T \arrow[d] \arrow[dr] \arrow[rr, Rightarrow, "\alpha"'] & & \Psi \arrow[dl] \\
\mathrm{FinSet}_0 & \mathrm{FinSet}_0 &
\end{tikzcd}$$

We draw an object $\psi \in \Psi$ as follows:



(this finite set is $p(\psi)$)

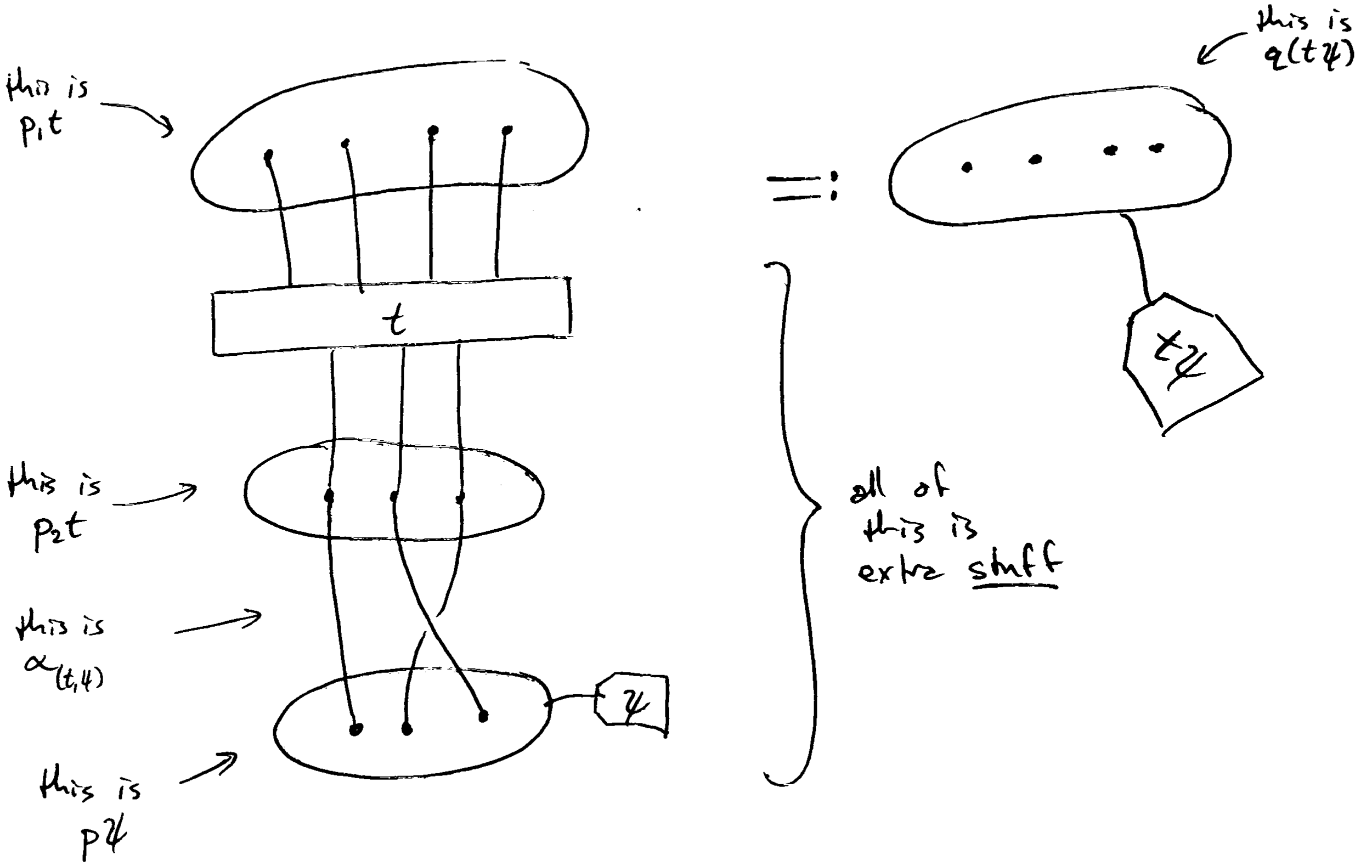
We draw an object $t \in T$ as follows:



(this set is $p_1 t$)

(this set is $p_2 t$)

An object of $T\Psi$ is then drawn as follows: we draw an object $t \in T$ and an object $\psi \in \Psi$ together with an isomorphism $\alpha : p_2 t \longrightarrow pt$

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<u>Example</u>: The Annihilation stuff operator, $A$.

This goes as follows

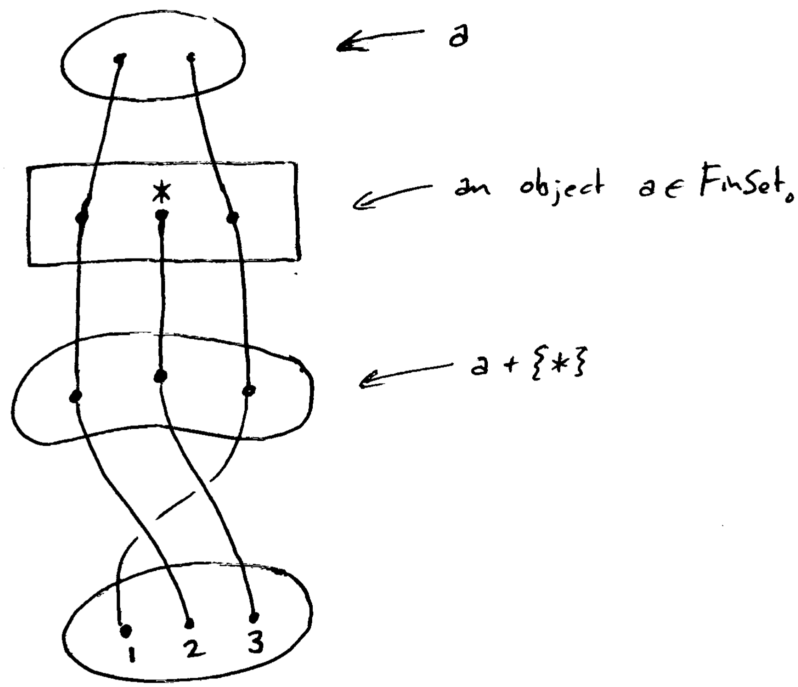
$$\begin{tikzcd} & \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "+\{*\}"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

where the right hand functor does

$$S \longmapsto S + \{*\}$$

to any $S \in \mathrm{FinSet}_0$. Let's apply this stuff operator to the stuff type $Z^n$ = "being a totally ordered n-elt set."

So if $n = 3$




An object of $AZ^3$ is a 2-element set with a total ordering on $S + 1$.

So: $$AZ^3 = 3Z^2$$

since there are 3 positions for "$*$" in the total order on $S + \{*\}$