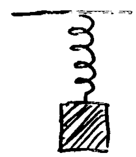
Stuff Operators, Perturbation Theory & Feynman Diagrams

We started this course by talking about a rock on a spring:



taking quantum mechanics into account! So far, we've focused on the case of an "ideal" spring, satisfying Hooke's law exactly:

$$\text{Force} \propto \text{displacement}$$

After a suitable choice of units, this gives us this formula for the rock's energy

$$H_0 = \frac{1}{2}(p^2 + q^2 - 1) \qquad \text{Harmonic Oscillator Hamiltonian}$$

To see how the rock oscillates, we start it off in some state ψ & solve Schrödinger's equation:

$$i \frac{d\psi(t)}{dt} = H_0 \psi(t)$$

with initial conditions $\psi(0) = \psi$. We can solve this in "closed form" since Fock space (the Hilbert space of states) has a nice basis of eigenvectors of H_0, namely z^n:

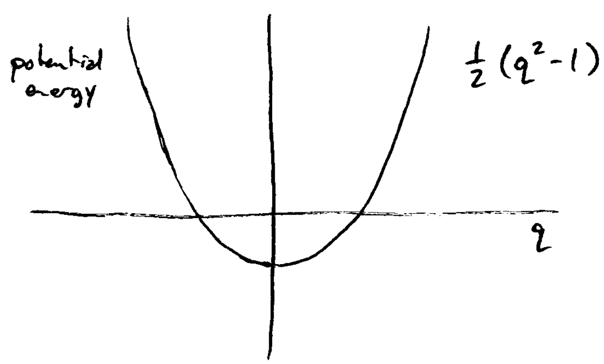
$$H_0 z^n = n z^n$$
$$\langle z^n, z^m \rangle = \delta_{nm} n!$$

If $\psi = \sum_{n \in \mathbb{N}} a_n z^n$, then we just get

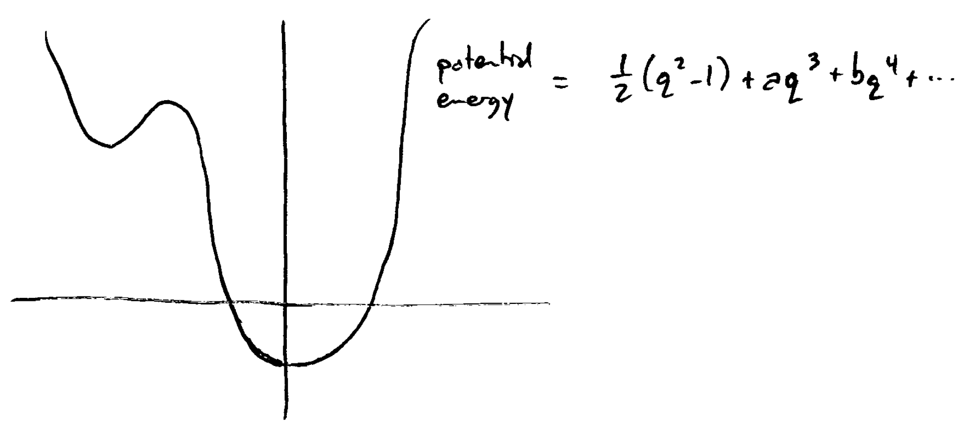
$$\psi(t) = e^{-itH_0}\psi$$
$$= \sum_{n \in \mathbb{N}} e^{-itn} a_n z^n$$

(This is what makes the harmonic oscillator so nice to work with!)

Now suppose we have a general oscillator, e.g. a spring that only satisfies Hooke's law approximately. Instead of our nice quadratic potential:



we have something like:



assuming potential is analytic & has a nondegenerate local minimum (nonvanishing 2nd derivative) — and we can place this minimum to coincide w/ the one for the harmonic oscillator, after suitable fiddling w/ coords.

To do perturbation theory, we'll write the Hamiltonian for this oscillator as

$$H = H_0 + V$$

where H_0 is the harmonic oscillator hamiltonian and

$$V = aq^3 + bq^4 + \cdots$$

In practice we'll often take V to be a polynomial. Now let's solve Schrödinger's equation perturbatively. We want to solve

$$i \frac{d\psi(t)}{dt} = H\psi$$

with initial conditions

$$\psi(0) = \psi.$$

The solution is

$$\psi(t) = e^{-itH}\psi$$

but how do we calculate e^{-itH}? This is hard, but e^{-itH_0} is easy, so we hope

$$e^{-itH}\psi \approx e^{-itH_0}\psi$$

$$\text{or} \quad e^{itH_0}e^{-itH}\psi \approx \psi$$

This expression:

$$e^{itH_0}e^{-itH}\psi =: \psi_{int}(t)$$

note: this says we evolve ψ forward in time by t using H and then backward in time by the same amount using H_0, so we are "factoring out" the time dependence of H_0

should therefore change slowly when H is close to H_0.

So let's calculate $\psi_{int}(t)$ & then

$$\begin{aligned}\psi(t) &= e^{-itH}\psi \\ &= e^{-itH_0}\psi_{int}(t)\end{aligned}$$

will be easy to calculate since we have a closed form for e^{-itH_0}. So let's turn our attention to $\psi_{int}(t)$. Let's find the differential equation it satisfies (secretly Schrödinger's equation) & solve it.

$$\begin{aligned}\frac{d}{dt}\psi_{int}(t) &= \frac{d}{dt}\, e^{itH_0}\overbrace{e^{-itH}\psi}^{\psi(t)} \\ &= e^{itH_0}\, iH_0\, e^{-itH}\psi + e^{itH_0}(-iH)e^{-itH}\psi \\ &= e^{itH_0}\, i(H_0 - H)\, e^{-itH}\psi \\ &= -i\, e^{itH_0} V e^{-itH_0}\psi_{int}(t) \\ &= -i\, V_{int}(t)\,\psi_{int}(t)\end{aligned}$$

Note: $e^{-itH}\psi = e^{-itH_0}\psi_{int}(t)$

where

$$V_{int}(t) = e^{itH_0} V e^{-itH_0}$$

is the potential V measured at time t in harmonic oscillator, (in the Heisenberg representation). In short, we need to solve Schrödinger's equation in the <u>interaction representation</u>:

$$\frac{d\psi_{int}(t)}{dt} = -i\, V_{int}(t)\,\psi_{int}(t) \qquad \text{w/ } \psi_{int}(0) = \psi$$

How do we solve it? ...

Integrate both sides!

$$\psi_{int}(t) = -i\int_0^t V_{int}(t_0)\,\psi_{int}(t_0)\,dt_0 + \psi$$

The only problem is we don't know $\psi_{int}(t_0)$. But don't worry, we have a formula for it — namely, this very formula! Using this, we get:

$$\psi_{int}(t) = -i\int_0^t V_{int}(t_0)\left[-i\int_0^{t_0} V_{int}(t_1)\,\psi_{int}(t_1)\,dt_1 + \psi\right]dt_0 + \psi$$

$$= \psi + \int_0^t i^{-1}\, V_{int}(t_0)\,\psi\, dt_0 + \int_0^t\int_0^{t_0} i^{-2}\, V_{int}(t_0)\, V_{int}(t_1)\,\psi_{int}(t_1)\,dt_1\,dt_0$$

This is great — the only problem is it contains $\psi_{int}(t_1)$. But we have a formula for it, namely this formula! So turn the crank again... forever.

We get:

$$\psi_{int}(t) = \sum_{n=0}^{\infty} \int_0^t\int_0^{t_0}\int_0^{t_1}\cdots\int_0^{t_{n-2}} i^{-n}\, V_{int}(t_0)V_{int}(t_1)\cdots V_{int}(t_{n-1})\,\psi\, dt_{n-1}\cdots dt_0$$

$$= \sum_{n=0}^{\infty} \int_{t\ge t_0\ge\cdots\ge t_{n-1}\ge 0} i^{-n}\, V_{int}(t_0)\cdots V_{int}(t_{n-1})\,\psi\, dt_{n-1}\, dt_0$$

as in the homework for week 5 (up to some notational differences in dummy variables).

If you feel funny about this solution, differentiate it & show:

$$i\,\frac{d\psi_{int}(t)}{dt} = -V_{int}(t)\,\psi_{int}(t)$$

13 May 2004

Last time we "showed" (nonrigorously but very generally) that if

$$H = H_0 + V$$

Hamiltonian — "free Hamiltonian" — "interaction Hamiltonian"

then the Schrödinger equation

$$i \frac{d\psi(t)}{dt} = H\psi(t) \qquad \psi(0) = \psi$$

can be solved by setting

$$\psi_{int}(t) = e^{itH_0}\psi(t)$$

$$V_{int}(t) = e^{itH_0} V e^{-itH_0}$$

& seeing that

$$\psi_{int}(t) = \sum_{n=0}^{\infty} (-i)^n \int_{0 \le t_1 \le \cdots \le t_n \le t} V_{int}(t_n) \cdots V_{int}(t_1)\psi \; dt_1 \cdots dt_n$$

so

$$\psi(t) = e^{-itH_0}\psi_{int}(t)$$

$$= \sum_{n=0}^{\infty} (-i)^n \int_{0 \le t_1 \le \cdots \le t_n \le t} e^{-i(t-t_n)H_0} V e^{-i(t_n - t_{n-1})H_0} V \cdots V e^{-i(t_1 - 0)H_0} \psi \; dt_1 \cdots dt_n$$

We are interested in the example where H_0 is the Hamiltonian of the harmonic oscillator & $V = V(q)$ is some polynomial in q. In the homework, we considered the case $V(q) = q = \frac{a+a^*}{\sqrt{2}}$. Let's take $\psi = z^\ell \in \mathbb{C}[z]$ & recall:

$$H_0 z^\ell = \ell z^\ell$$

— i.e. z^ℓ (after normalization) is the state of the harmonic oscillator with energy ℓ, or <u>ℓ "quanta" of energy</u> (we want to emphasize this perspective because we are interested in categorification!) Thus,

$$e^{-itH_0} z^\ell = e^{-it\ell} z^\ell$$

So time evolution according to the <u>free</u> Hamiltonian is simple — it just multiplies z^ℓ by a phase, which really means nothing detectable is happening.

So in this expression

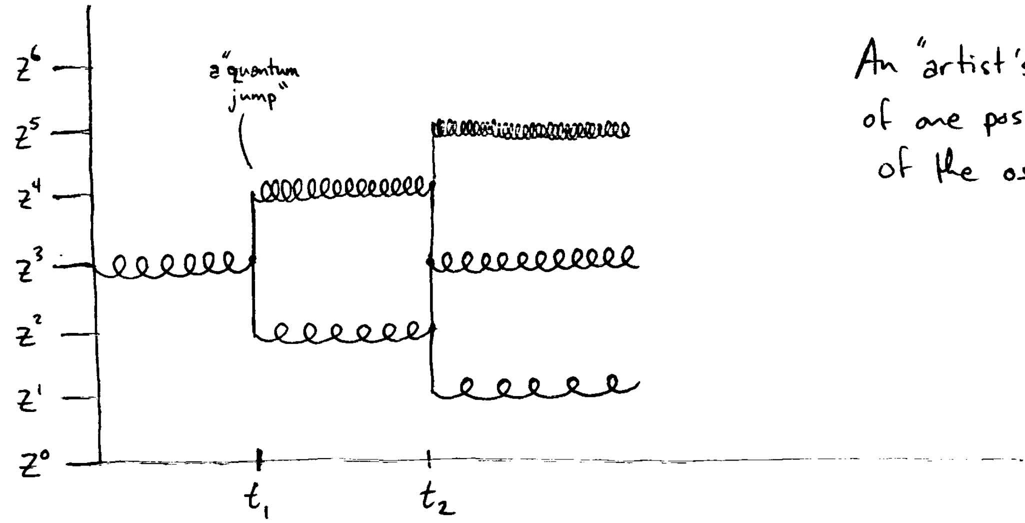
$$e^{-i(t-t_n)H_0}\left(\frac{a+a^*}{\sqrt{2}}\right)e^{-i(t_n-t_{n-1})}\left(\frac{a+a^*}{\sqrt{2}}\right)\cdots\left(\frac{a+a^*}{\sqrt{2}}\right)e^{-it_1H_0} z^\ell$$

the state z^ℓ only changes <u>phase</u> until time t_1. Then $\left(\frac{a+a^*}{\sqrt{2}}\right)$ acts on it:

$$\left(\frac{a+a^*}{\sqrt{2}}\right)z^\ell = \frac{1}{\sqrt{2}}\left(\ell z^{\ell-1} + z^{\ell+1}\right)$$

so it gains or loses one quantum of energy. And so on...

... giving a picture like this

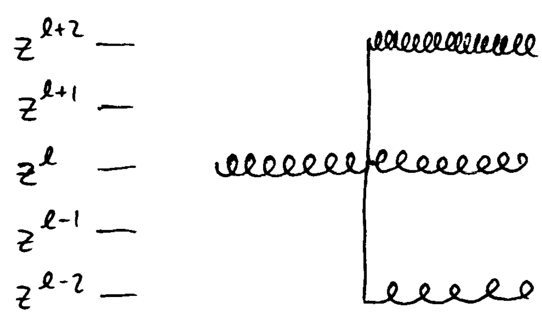


An "artist's" depiction of one possible history of the oscillator.

For more complicated V, the picture gets fancier, e.g.:

$$V(q) = q^2 = \left(\frac{a+a^*}{\sqrt{2}}\right)^2 = \frac{a^2 + aa^* + a^*a + a^{*2}}{2}$$

gives quantum jumps that look like:



These pictures aren't as useful as the "Feynman diagrams" we'll soon draw, where Z^ℓ will be drawn as a totally ordered set of ℓ dots — since Z^ℓ = "being a totally ordered ℓ-elt set"

These dots represent "quanta of energy."

Now let's calculate transition amplitudes:

$$\langle \varphi, \psi(t) \rangle = \langle \varphi, e^{-itH} \psi \rangle$$

(so that $|\langle \varphi, e^{-itH}\psi \rangle|^2$ is the probability of finding the system in state φ at time t given that it was put in state ψ at time 0) It suffices to compute these transition amplitudes for a basis:

$$\langle z^k, e^{-itH} z^\ell \rangle =$$

$$\sum_{n=0}^{\infty} (-i)^n \int_{0 \le t_1 \le \cdots \le t_n \le t} \langle z^k, e^{-i(t-t_n)H_0} V e^{-i(t_n - t_{n-1})H_0} V \cdots V e^{-it_1 H_0} z^\ell \rangle dt_1 \cdots dt_n$$

To do this, let's focus on the inner product. In fact, we'll warm up by considering the simple case $0 = t_1 = t_2 = \cdots = t_n = t$, i.e.

$$\langle z^k, V^n z^\ell \rangle$$

For starters, let's try the case

$$V = q = \frac{a + a^*}{\sqrt{2}}$$

But to reduce the clutter, let's actually take V to be $\sqrt{2} q$, which we'll call

$$\varphi = a + a^* \qquad \text{the "field operator".}$$

(Avoiding $\sqrt{2}$ will make categorification much easier!)

So: calculate

$$\langle z^k, \varphi^n z^\ell \rangle$$

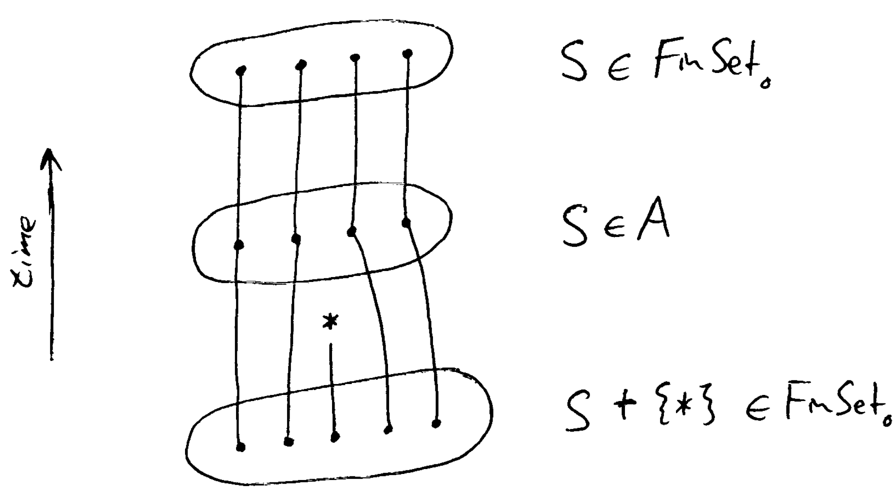
To do this, we'll use the stuff types Z^k, Z^ℓ, the inner product of stuff types, and some stuff operator

$$\Phi = A + A^*$$

What's this? We've already discussed the "annihilation stuff-operator" A:

$$\begin{tikzcd} & A = \mathrm{FinSet}_0 \arrow[dl, "1"'] \arrow[dr, "\_+\{*\}"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

Recall that we draw an object of A as:



This is our first Feynman diagram! (such a simple one you'll never see it in any physics book...)

Next, what's A^*? It's a "time reversed version" of A. More precisely, given any stuff operator T:

$$\begin{tikzcd} & T \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ FinSet_0 & & FinSet_0 \end{tikzcd}$$

we define its <u>adjoint</u> T^* to be the stuff operator:

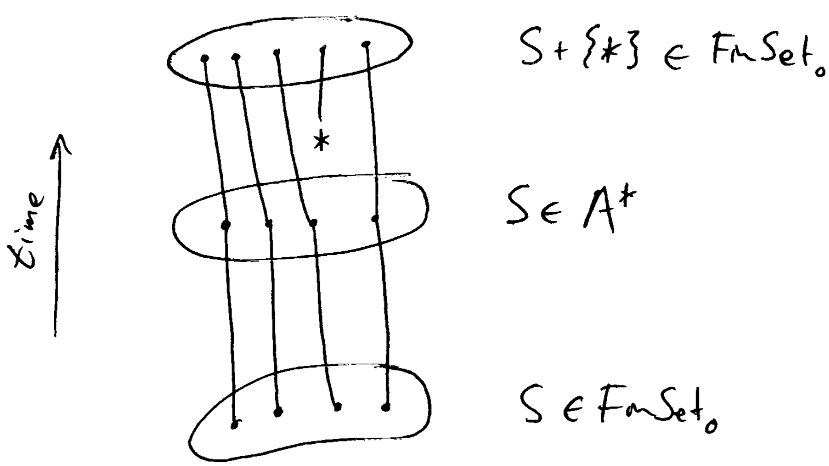
$$\begin{tikzcd} & T^* = T \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ FinSet_0 & & FinSet_0 \end{tikzcd}$$

— like transposing a matrix

In particular, A^* will be:

$$\begin{tikzcd} & A^* = FinSet_0 \arrow[dl, "\_ + \{*\}"'] \arrow[dr, "1"] & \\ FinSet_0 & & FinSet_0 \end{tikzcd}$$

So an object of A^* can be drawn like



Next, what's $\Phi = A + A^*$? How do we add stuff operators? Given two stuff operators S & T:

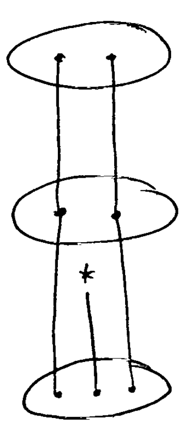
$$\begin{tikzcd} & S \arrow[dl, "p_1^S"'] \arrow[dr, "p_2^S"] & \\ FinSet_0 & & FinSet_0 \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl, "p_1^T"'] \arrow[dr, "p_2^T"] & \\ FinSet_0 & & FinSet_0 \end{tikzcd}$$

their sum is

$$\begin{tikzcd} & S+T \arrow[dl, "p_1^S + p_1^T"'] \arrow[dr, "p_2^S + p_2^T"] & \\ FinSet_0 & & FinSet_0 \end{tikzcd}$$

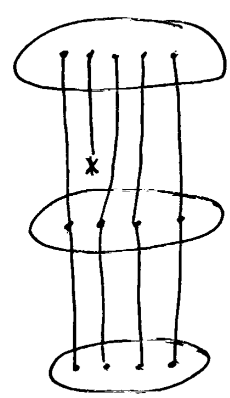
(with $p_i^S + p_i^T$ defined in the obvious way — an object of $S+T$ is either an object of S or an object of T, so $p_1^S + p_1^T$ acts as p_1^S or p_1^T as appropriate)

For example, an object of $\Phi = A + A^*$ will look like



(an object of A)

<u>or:</u>



(an object of A^*)