18 May 2004

Stuff Operators, Perturbation Theory
& Feynman Diagrams (cont.)

Our goal: to compute $\langle z^k, \varphi^n z^\ell \rangle$ where $z^k, z^\ell \in \mathbb{C}[z]$
& $\varphi : \mathbb{C}[z] \to \mathbb{C}[z]$ is given by

$$\varphi = \sqrt{2}\, q = a + a^*$$

We'll do this by computing this categorified version

$$\langle Z^k, \Phi^n Z^\ell \rangle$$

& then using this (unproved) fact:

$$|\langle Z^k, \Phi^n Z^\ell \rangle| = \langle |Z^k|, |\Phi^n Z^\ell| \rangle$$
$$= \langle z^k, \varphi^n z^\ell \rangle$$

(It turns out the <u>easiest</u> way to calculate this number is by this method — Physicists actually do it this way without knowing about categorification, i.e. they do it with Feynman diagrams)

How can we describe a groupoid like $\langle Z^k, \Phi^n Z^\ell \rangle$ or more generally $\langle \Psi, T\Psi' \rangle$ where Ψ, Ψ' are stuff types & T is a stuff operator? We'll draw a picture of a typical object in $\langle \Psi, T\Psi' \rangle$ — this picture will be (an abstract version of) a Feynman diagram. To start with, let's draw a typical object $\psi' \in \Psi'$. Remember Ψ' is a stuff type

$$\begin{tikzcd} \Psi' \arrow[d, "p"] \\ \mathrm{FinSet}_0 \end{tikzcd}$$

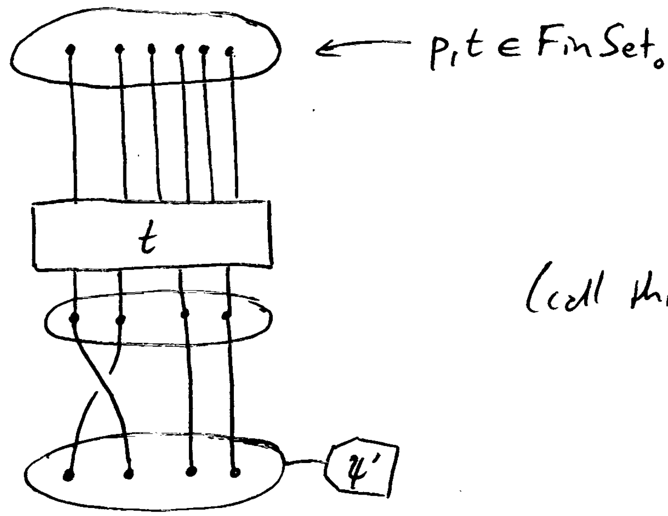
So ψ' is a "finite set equipped w. extra Ψ'-stuff."

$p\psi' \in \mathrm{FinSet}_0$

Next let's draw a typical object of $T\Psi'$. Remember $T\Psi'$ is defined as the weak pullback

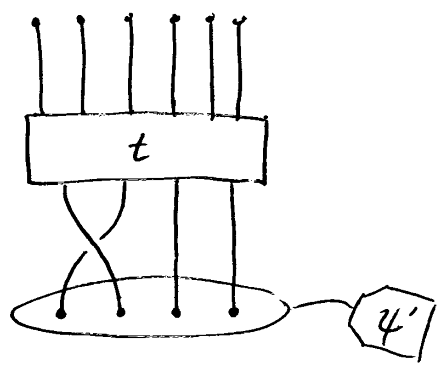
$$\begin{tikzcd} & & T\Psi' \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "q"'] \arrow[dd, phantom, "\vee" near start] & \\ & T \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \Psi' \arrow[dl, "p'"] \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 & \end{tikzcd}$$

So an object of $T\Psi'$ consists of an object $t \in T$, an object $\psi' \in \Psi'$ & an isomorphism $\alpha : p_2 t \xrightarrow{\sim} p'\psi'$. We draw this as:



(call this $t\psi'$)

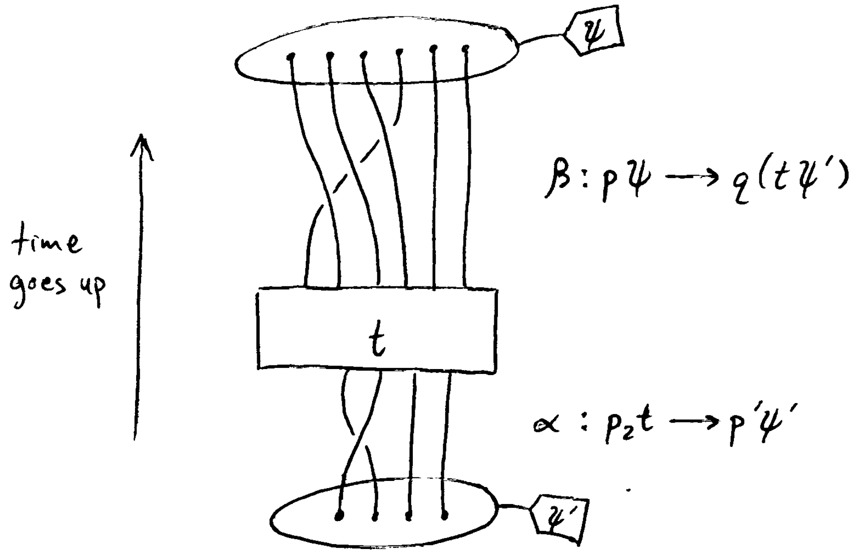
Or, we can remove some of the clutter & get a little closer to Feynman diagrams:



Next let's draw a typical object of $\langle \Psi, T\Psi' \rangle$. Remember $\langle \Psi, T\Psi' \rangle$ is defined as the weak pullback:

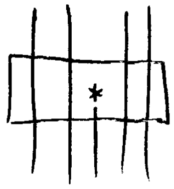
$$\begin{tikzcd}
 & \langle \Psi, T\Psi' \rangle \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & \\
\Psi \arrow[dr, "p"'] & & T\Psi' \arrow[dl, "q"] \\
 & FinSet_0 &
\end{tikzcd}$$

So an object of $\langle \Psi, T\Psi' \rangle$ consists of an object $\psi \in \Psi$, an object $t\psi' \in T\Psi'$ & an iso $\beta : p\psi \to q(t\psi')$. We draw this as:



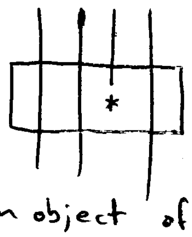
Now let's see what this looks like when $T = \Phi^n$.

Recall that an object of $\Phi = A + A^*$ looks like either:



an object of A

or

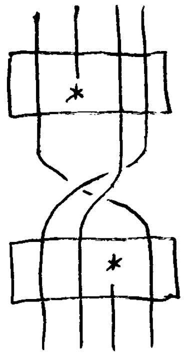


an object of A^*.

Recall that Φ^2 is defined using composition of stuff operators, namely this weak pullback:

$$\begin{tikzcd}
 & & \Phi^2 \arrow[dl] \arrow[dr] & & \\
 & \Phi \arrow[dl] \arrow[dr] & & \Phi \arrow[dl] \arrow[dr] & \\
\mathrm{FinSet}_0 & & \mathrm{FinSet}_0 & & \mathrm{FinSet}_0
\end{tikzcd}$$

so a typical object looks like



← the isomorphism in def. of weak pullback.

Note: $\Phi^2 = (A + A^*)(A + A^*) = A^2 + AA^* + A^*A + A^{*2}$

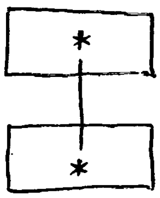
so we get four kinds of objects in Φ^2: the one drawn above is an object of A^*A.

Now let's calculate

$$\langle 1, \Phi^2 1 \rangle.$$

We'll draw a typical object of this groupoid. Here Z^n is "being a totally ordered n-elt. set" so 1 is "being the empty set." So the only object of $\langle 1, \Phi^2 1\rangle$ is:

← the empty set



← the empty set

So: $\langle 1, \Phi^2 1 \rangle \simeq 1$, the groupoid with one object & only the identity morphism (one should check the last bit!)

So: $\langle 1, \varphi^2 1 \rangle = |\langle 1, \Phi^2 1\rangle| = |1| = 1$.

Another way to show this:

$$\langle 1, \varphi^2 1\rangle = \langle 1, (a+a^*)(a+a^*)1\rangle$$

$$= \langle 1, a^2 1\rangle + \langle 1, a^* a 1\rangle + \langle 1, a a^* 1\rangle + \langle 1, a^{*2} 1\rangle$$

$\langle 1, a^2 1\rangle = 0$ since $a1=0$

$\langle 1, a^* a 1\rangle = 0$ since $a1 = 0$

$\langle 1, a a^* 1\rangle = \langle a^* 1, a^* 1\rangle = \langle z, z\rangle = 1$ since $\langle z^k, z^l\rangle = k!\,\delta_{kl}$

$\langle 1, a^{*2} 1\rangle = \langle 1, z^2\rangle = 0$

Note: the term that gives 1 corresponds to the picture we drew ("create, then annihilate").

Another example: the vacuum expectation value of Φ^4:

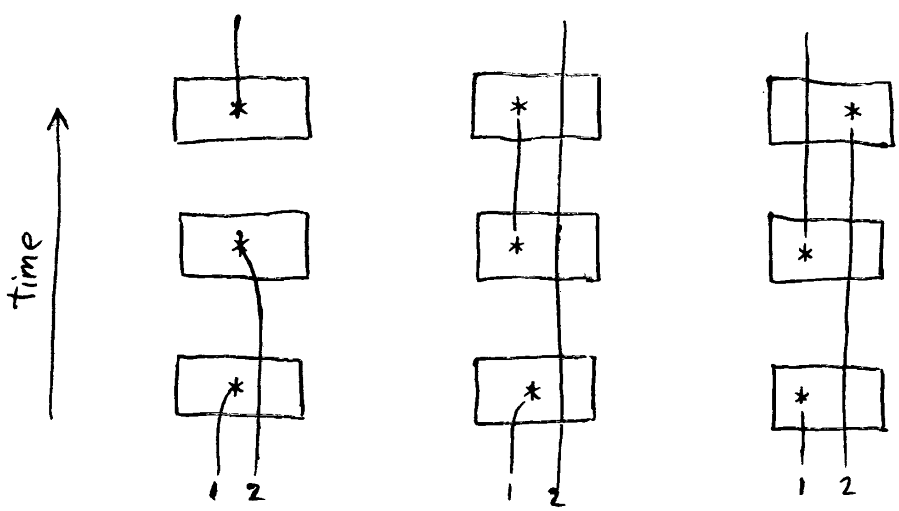
$$\langle 1, \Phi^4 1\rangle$$

on second thought, that's part of the homework.

Let's do:

$$\langle Z, \Phi^3 Z^2\rangle$$

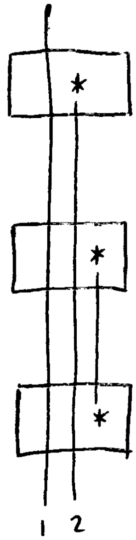
Objects in here look like



(an object in $A^* A A \subseteq \Phi^3$)

(these two are objects in $A A^* A \subseteq \Phi$)

and ones similar to these where particle #2 is annihilated (first) instead of #1



(This is an object in AAA^* but there are more...)

$$|\langle Z, \Phi^3 Z^2\rangle| = \langle z, \varphi^3 z^2\rangle$$

is the cardinality of this groupoid: sum over iso. classes of objects $[x]$ of $\frac{1}{\mathrm{Aut}(x)}$.

---

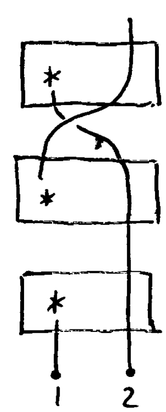
20 May 2004

Feynman Diagrams

Last time we were calculating the groupoid

$$\langle Z, \Phi^3 Z^2\rangle$$

We draw the objects like this:

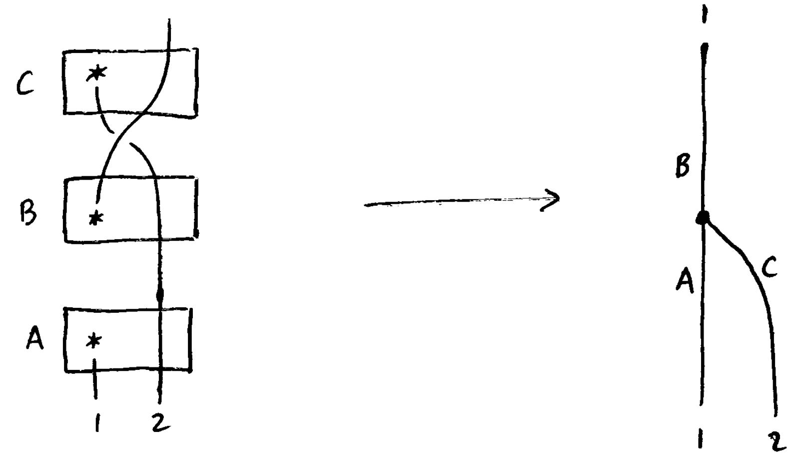


C

B

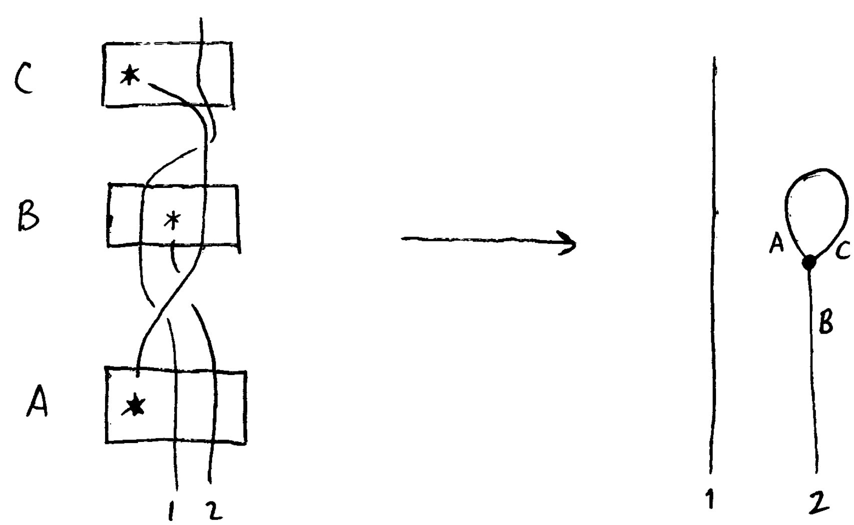
A

but it was tiresome to count all such objects. They'll be easier to keep track of if we draw all the $*$'s as a single dot, or vertex - but then label all the edges incident to this vertex by A, B, & C, to keep track of which $*$ was in which box: box A, B, or C. (The "boxes" correspond to applications of Φ & thus are totally ordered). So



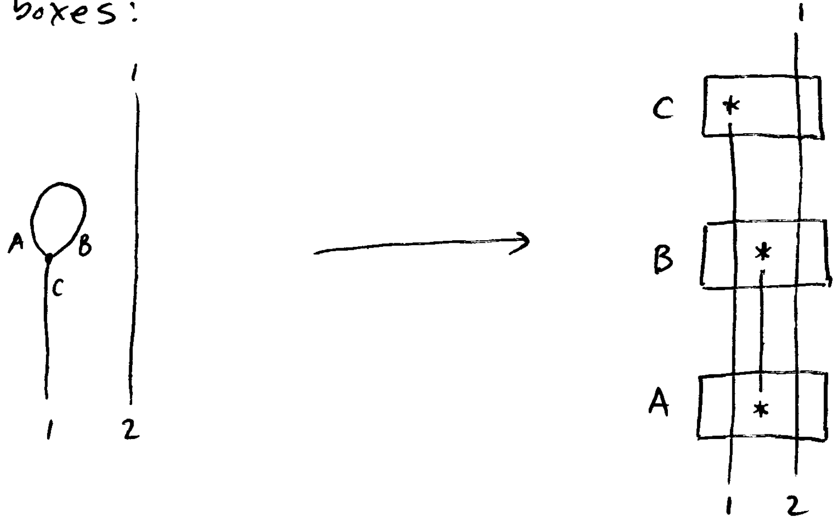
a Feynman diagram!

To see that we understand this, let's translate another one of our old diagrams into a Feynman diagram.

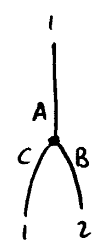


Note: It's not really the edges that get labelled A, B, & C but the incidences

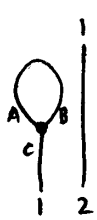
Let's also practice translating back: take any Feynman diagram with one trivalent vertex with incidences labelled A, B, & C, two incoming edges (totally ordered) and one outgoing edge (totally ordered) and draw it using boxes:



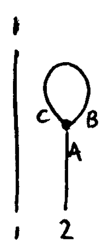
Using Feynman diagrams, we can easily list (or count) all the objects in (a skeletal version of) the groupoid $\langle Z, \Phi^3 Z^2 \rangle$:



& five more, corr. to 3! total orderings of the "incidences"



& two more, corr. to labellings of the incoming edge (note: by symmetry  $\cong$ )



& two more, corr. to labellings of the incoming edge.

The only automorphisms (symmetries) of these objects are the identity, so the groupoid $\langle Z, \Phi^3 Z^2 \rangle$ is discrete (really just a set):

$$\langle Z, \Phi^3 Z^2 \rangle \simeq 12$$

so

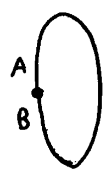
$$|\langle Z, \Phi^3 Z^2 \rangle| = \langle z, \varphi^3 z^2 \rangle = 12$$

(12 = 6 ×  + 3 ×  + 3 × )

Let's do some more!

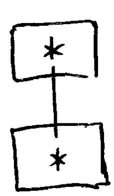
<u>Example:</u> $\langle 1, \Phi^2 1 \rangle$ is a groupoid with object

the empty set ⟶

bivalent vertex: Φ^2 ⟶ 

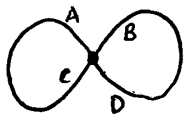
the empty set ⟶

or in the old style:



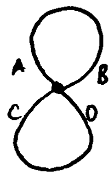
There's only one object (in skeletal version) & only identity morphisms, so $\langle 1, \Phi^2 1 \rangle \simeq 1$.

<u>Example:</u> $\langle 1, \Phi^4 1 \rangle$. Objects in here look like

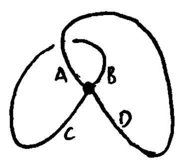


4-valent vertex
since Φ^4

or:

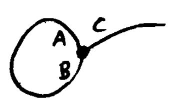


or



& that's all. So $\langle 1, \Phi^4 1 \rangle \simeq 3$

<u>Example:</u> $\langle 1, \Phi^3 1 \rangle \simeq 0$, since 3 is an odd number, so there's no perfect matching between incidences

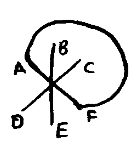


?

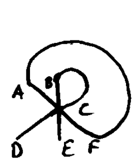
In general:

$$\langle 1, \Phi^n 1 \rangle = \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & \end{cases}$$

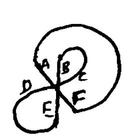
since when n is even, there are $n-1$ ways to match A:



& $n-3$ ways to make the next choice:



& so on

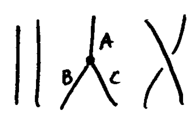


, for a total of $(n-1)!!$

So now we know

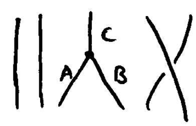
$$\langle 1, \underbrace{(a+a^*)\cdots(a+a^*)}_{n \text{ even}} 1 \rangle = (n-1)!!$$

Now, let's look at the stuff operator $\frac{\Phi^n}{n!}$. Here we use the fact that the group $n!$ acts on Φ^n by permuting the labels $A, B, C, \ldots$ of the incidences: e.g. the permutation $\begin{pmatrix} A & B & C \\ \downarrow & \downarrow & \downarrow \\ C & A & B \end{pmatrix}$ maps the object



$\in \Phi^3$

to



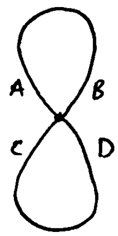
$\in \Phi^3$

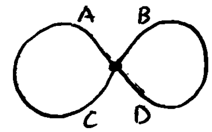
So we can define the weak quotient $\frac{\Phi^n}{n!}$ & get a stuff operator:

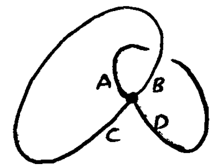
$$\begin{tikzcd} & \Phi^n /\!/ n! \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{FinSet}_0 & & \mathrm{FinSet}_0 \end{tikzcd}$$

Example: $\langle 1, \frac{\Phi^4}{4!} 1 \rangle$

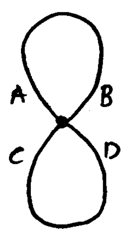
Now we have 3 objects as before







but there are new isomorphisms corr. to label-changing! So in fact all objects are isomorphic, so in a skeleton we pick one:



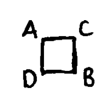
but it has lots of automorphisms — symmetries,

e.g. $\begin{matrix} A & B & C & D \\ \downarrow & \downarrow & \downarrow & \downarrow \\ B & A & C & D \end{matrix}$ $\quad$ $\begin{matrix} A & B & C & D \\ \downarrow & \downarrow & \downarrow & \downarrow \\ A & B & D & C \end{matrix}$ (these two generate the Klein 4-group)

and $\begin{matrix} A & B & C & D \\ \downarrow & \downarrow & \downarrow & \downarrow \\ D & C & B & A \end{matrix}$ (rotation)

These 3 generate an 8-elt. group

it's D_4, the symmetry gp. of the square with vertices labelled as:



So: $$\langle 1, \frac{\varphi^4}{4!} 1 \rangle = \left| \langle 1, \frac{\Phi^4}{4!} 1 \rangle \right| = \frac{1}{8}$$

which is consistent with

$$\langle 1, \frac{\varphi^4}{4!} 1 \rangle = \frac{1}{4!} \langle 1, \varphi^4 1 \rangle = \frac{1}{4!} \left| \langle 1, \Phi^4 1 \rangle \right| = \frac{3}{4!} = \frac{1}{8}.$$