Wick Powers

Summing over all Feynman diagrams gets a lot easier if we use the "Wick powers" or "normal-ordered powers" of the field operator

$$\varphi = a + a^*$$

These Wick powers $:\varphi^n:$ are just an alternate basis of the space of polynomials in φ, defined by

$$\forall n>0 \quad \langle 1, :\varphi^n: 1 \rangle = 0 \quad (\text{as opposed to } \langle 1, \varphi^n, 1\rangle \neq 0)$$

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$$[a, :\varphi^n:] = n :\varphi^{n-1}: \qquad (\text{like } [a, \varphi^n] = n\varphi^{n-1})$$

$$[a^*, :\varphi^n:] = -n :\varphi^{n-1}: \qquad (\text{like } [a^*, \varphi^n] = -n\varphi^{n-1})$$

and $$:\varphi^0: = \varphi^0 = 1$$

In homework we instead defined $:q^n: = \dfrac{:\varphi^n:}{(\sqrt{2})^n}$ by requiring

$$\langle 1, :q^n: 1 \rangle = 0 \quad \text{for } n>0$$

$$[q, :q^n:] = 0$$

$$[ip, :q^n:] = n :q^{n-1}:$$

$$:q^0: = q^0 = 1$$

This is clearly equivalent if we use

$$a = \frac{q+ip}{\sqrt{2}} \qquad a^* = \frac{q-ip}{\sqrt{2}}.$$

The idea in either case is to make q^n (or φ^n) behave "more classically": since $q^n = 0$ in the lowest-energy state of the classical oscillator, we want the ground-state expectation value $\langle 1, q^n 1\rangle$ to be zero, which we achieve by replacing q^n by $:q^n:$.

In homework we saw

$$:q^n: = \left(\frac{1}{\sqrt{2}}\right)^n \sum_{0\le k\le n} \binom{n}{k} a^{*k} a^{n-k}$$

i.e. it's like q^n, but we brutally push all the annihilation operators to the right to ensure

$$\langle 1, :q^n: 1\rangle = 0 \quad \text{for } n>0$$

It's not obvious, but true, that $:q^n:$ is a polynomial in q.

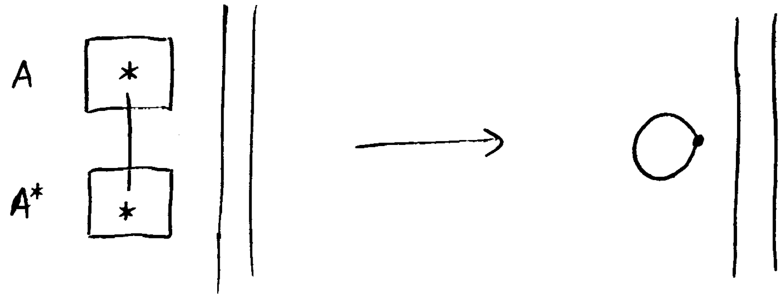
Now let's turn to

$$:\varphi^n: = \sum_{0\le k\le n} \binom{n}{k} a^{*k} a^{n-k}$$

and the stuff operator

$$:\Phi^n: = \sum_{0\le k\le n} \binom{n}{k} A^{*k} A^{n-k}$$

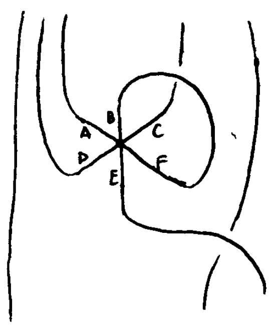
& describe this stuff operator using Feynman diagrams. The big difference between using this and using Φ^n is that there are no <u>loops</u>, since these arise from terms with AA^* in them:



This makes keeping track of Feynman diagrams a lot easier!

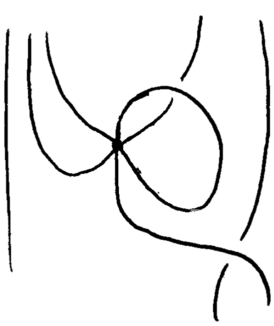
Four Stuff Operators:

Φ^n



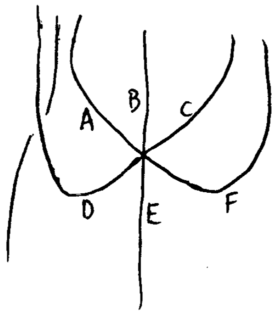
Feynman diagrams with one n-valent vertex with totally ordered incidences.

$\dfrac{\Phi^n}{n!}$



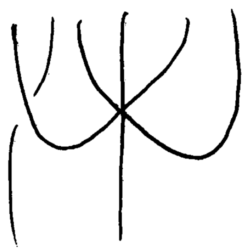
Feynman diagrams with one n-valent vertex.

$:\Phi^n:$



Feynman diagrams with one n-valent vertex with totally ordered incidences and no loops.

$\dfrac{:\Phi^n:}{n!}$

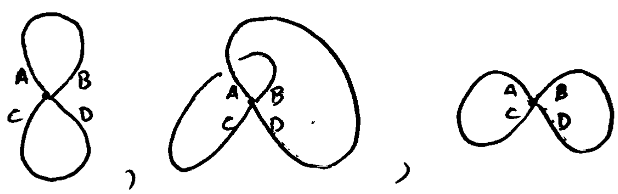


Feynman diagrams with one n-valent vertex and no loops.

Examples:

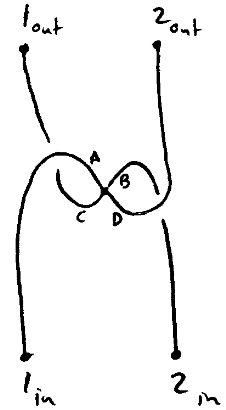
1) $\langle 1, \Phi^n 1\rangle = \begin{cases} (n-1)!! & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

& we've seen the nonzero values all come from diagrams with loops;

$\langle 1, \Phi^4 1\rangle = \Big\{$  $\Big\} \simeq 3$

Thus $\langle 1, :\Phi^n: 1\rangle = 0$ since this is the subset of $\langle 1, \Phi^n 1\rangle$ consisting of diagrams with no loops — but there aren't any!

2) $\langle Z^2, :\Phi^4: Z^2 \rangle$ has objects like



$\times 4 \quad \times 3 \quad \times 2 \quad \times 1 = 4!$

$\times 4$: 1st incoming particle on edge A, B, C, or D

$\times 3$: 2nd incoming

$\times 2$: 1st outgoing

$\times 1$: 2nd outgoing

i.e., we're getting the set of bijections

$$f: \{A, B, C, D\} \longrightarrow \{1_{in}, 2_{in}, 1_{out}, 2_{out}\}$$

(There are no other objects, since they would have loops!)

Or:

$$|\langle Z^2, :\Phi^4: Z^2 \rangle| = \langle z^2, :\varphi^4: z^2 \rangle$$

where

$$:\varphi^4: = a^4 + 4a^*a^3 + 6a^{*2}a^2 + 4a^{*3}a + a^{*4}$$

This equals

$$\langle z^2, 6a^{*2}a^2, z^2 \rangle$$

since other terms vanish. This equals

$$6\langle a^2z^2, a^2z^2 \rangle = 6\langle 2, 2 \rangle = 24$$

Similarly:

$$\langle Z^n, :\Phi^{n+m}: Z^m \rangle = (n+m)!$$

(show it!)

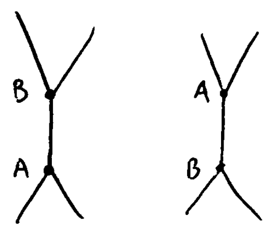
3) In homework, we saw

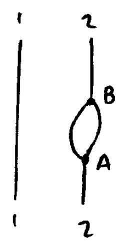
$$\left\langle Z^2, \frac{\Phi^3}{3!} \frac{\Phi^3}{3!} Z^2 \right\rangle$$

involved lots of Feynman diagrams – but most had loops. Now let's do

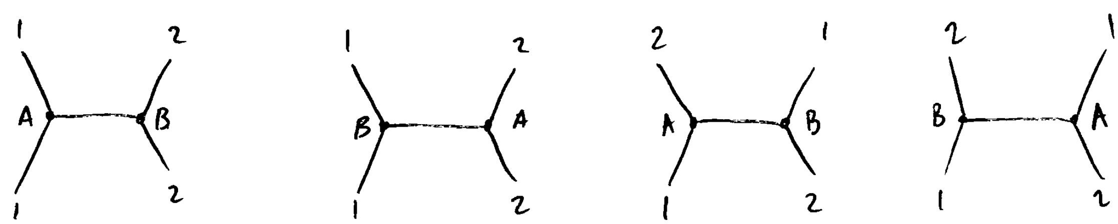
$$\left\langle Z^2, \frac{:\Phi^3:}{3!} \frac{:\Phi^3:}{3!} Z^2 \right\rangle$$

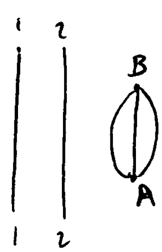
This has objects like





$\times \dfrac{2^3}{2}$





$\times \dfrac{2}{3!}$