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QUANTUM GRAVITY SEMINAR

Wick Powers

✓ $\frac{40}{40}$ GREAT!

1) An elt. of the Weyl algebra W is uniquely determined up to an additive constant by its commutators with p & q.

Proof: First we establish how $[ip, -]$ and $[-iq, -]$ act on elements f W. For monomials of the form $p^m q^n$, $m, n \in \mathbb{N}$, we have:

$$
\begin{aligned}
[ip, p^m q^n] &= ip^{m+1} q^n - i p^m q^n p \\
&= p^m (ipq^n - iq^n p) \\
&= p^m [ip, q^n] \\
&= \begin{cases} p^m n q^{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases} \qquad \left(= \text{“} \frac{\partial}{\partial q} p^n q^n \text{”}\right)
\end{aligned}
$$

$$
\begin{aligned}
[-iq, p^m q^n] &= -iqp^m q^n + ip^m q^{n+1} \\
&= (-iqp^m + ip^m q) q^n \\
&= [-iq, p^m] q^n \\
&= \begin{cases} mp^{m-1} q & m \geq 1 \\ 0 & m = 0 \end{cases} \qquad \left(= \text{“} \frac{\partial}{\partial p} p^m q^n \text{”}\right)
\end{aligned}
$$

Now we can get on with the proof. We assume w is of the form

$$w = \sum_{m,n \in \mathbb{N}} w_{mn} p^m q^n . \qquad w_{mn} \in \mathbb{C}, \text{ finitely many nonzero}$$

(Though we haven't proved w can be written like this, it seems obvious since we can always use the commutator to push all of the q's to the end of each term) We also use the fact that the coefficients w_{mn} are unique, i.e. that the $p^m q^n$ are linearly independent.

2) The Wick powers $:q^n: \in W$ (which will be constructed in #3) are uniquely determined by the conditions

$$[ip, :q^n:] = n :q^{n-1}: \qquad [q, :q^n:] = 0$$

$$\langle 1, :q^n: 1\rangle = 0 \quad \forall n>0$$

$$\text{and} \quad :q^0: = 1$$

Proof: Suppose $:q^n:$ and $:q^n:'$ are sequences of Wick powers of q. By part (1), since $[ip, :q^n: - :q^n:'] = 0$ and $[-iq, :q^n: - :q^n:'] = 0$, we have $:q^n: - :q^n:' = c \in \mathbb{C}$. For $n>0$ we then have

$$\begin{aligned} 0 &= \langle 1, :q^n: 1\rangle - \langle 1, :q^n:' 1\rangle \\ &= \langle 1, :q^n: - :q^n:' 1\rangle \\ &= \langle 1, c 1\rangle \\ &= c\langle 1, 1\rangle \\ &= c \end{aligned}$$

So $:q^n: = :q^n:'$ $\forall n>0$. Since $:q^0: = :q^0:'$ by definition, the Wick powers are all unique. ∎

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3) Let

$$:q^n: := \left(\frac{1}{\sqrt{2}}\right)^n \sum_{k=0}^{n} \binom{n}{k} a^{*k} a^{n-k}.$$

Then $:q^n:$ are Wick powers of q (which are unique by part (2))

Proof: We just need to check that $:q^n:$ so defined satisfy all of the properties listed in part (2).

In checking these properties we need these two commutators:

$$[a, a^{*n}] = n a^{*n-1} \quad \forall n \geq 1$$

Proof: For $n=1$, $[a, a^*] = 1$. Inductively, under the hypothesis $[a, a^{*n-1}] = (n-1)a^{*n-2}$ we get

$$\begin{aligned} a a^{*n} = (a a^{*n-1}) a^* &= ([a, a^{*n-1}] + a^{*n-1} a) a^* \\ &= (n-1) a^{*n-2} a^* + a^{*n-1} a a^* \\ &= (n-1) a^{*n-1} + a^{*n-1}(1 + a^* a) \\ &= n a^{*n-1} + a^{*n} a \end{aligned}$$

∎

$$[a^n, a^*] = n a^{n-1} \quad \forall n \geq 1$$

Proof: For $n=1$, $[a, a^*] = 1$. Inductively, under the hypothesis $[a^{n-1}, a^*] = (n-1)a^{n-2}$ we get

$$\begin{aligned} a^n a^* = a(a^{n-1} a^*) &= a([a^{n-1}, a^*] + a^* a^{n-1}) \\ &= a(n-1)a^{n-2} + a a^* a^{n-1} \\ &= (n-1) a^{n-1} + (1 + a^* a) a^{n-1} \\ &= n a^{n-1} + a^* a^n \end{aligned}$$

∎

(So $[a, -]$ acts as formal derivative w.r.t. a^*, while $[-, a^*]$ acts as formal derivative w.r.t. a)

Now on to checking the required properties ...

$$[q, :q^n:] = \left(\frac{1}{\sqrt{2}}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left[\frac{a+a^*}{\sqrt{2}}, a^{*k} a^{n-k}\right]$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \left([a, a^{*k}] a^{n-k} + a^{*k} [a^*, a^{n-k}] \right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} \left[\sum_{k=1}^{n} \binom{n}{k} k a^{*k-1} a^{n-k} - \sum_{k=0}^{n-1} \binom{n}{k} (n-k) a^{*k} a^{n-k-1} \right]$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} \left[\sum_{k=1}^{n} n \binom{n-1}{k-1} a^{*k-1} a^{n-k} - \sum_{k=0}^{n-1} n \binom{n}{k} a^{*k} a^{n-k-1} \right.$$

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$$\left. + \sum_{k=0}^{n-1} n \binom{n-1}{k-1} a^{*k} a^{n-k-1} \right]$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} n \sum_{k=0}^{n-1} \left(\binom{n-1}{k} a^{*k} a^{n-k-1} - \binom{n}{k} a^{*k} a^{n-k-1} + \binom{n-1}{k-1} a^{*k} a^{n-k-1} \right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} n \sum_{k=0}^{n-1} \left(\binom{n-1}{k} - \binom{n}{k} + \binom{n-1}{k-1} \right) a^{*k} a^{n-k-1}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} n \sum_{k=0}^{n-1} (0)\, a^{*k} a^{n-k-1} = 0$$

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1 $$k\binom{n}{k} = k \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{(k-1)!(n-k)!} = n \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = n \binom{n-1}{k-1}$$

2 $$\binom{n-1}{k} - \binom{n}{k} = \frac{(n-1)!}{k!(n-1-k)!} - \frac{n!}{k!(n-k)!} = \frac{(n-1)!(n-k)}{k!(n-k)!} - \frac{n!}{k!(n-k)!} = \frac{\cancel{n(n-1)!} - k(n-1)! - \cancel{n!}}{k!(n-k)!}$$

$$= \frac{-(n-1)!}{(k-1)!(n-k)!} = \frac{-(n-1)!}{(k-1)!((n-1)-(k-1))!} = -\binom{n-1}{k-1}$$

$$[ip, :q^n:] = \left(\frac{1}{\sqrt{2}}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left[\frac{a - a^*}{\sqrt{2}} , a^{*k} a^{n-k} \right]$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \left([a, a^{*k} a^{n-k}] - [a^*, a^{*k} a^{n-k}] \right)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \left([a, a^{*k}] a^{n-k} - a^{*k} [a^*, a^{n-k}] \right)$$

$$= \quad \vdots$$

The calculation is the same as the previous one except for this − sign so tracing through the calculation we see that we'll get:

$$\vdots$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} n \sum_{k=0}^{n} \left(\binom{n-1}{k} + \underbrace{\binom{n}{k} - \binom{n-1}{k-1}} \right) a^{*k} a^{n-1-k}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n+1} n \sum_{k=0}^{n} \left(\binom{n-1}{k} + \binom{n-1}{k} \right) a^{*k} a^{n-1-k}$$

by ② again.

$$= \left(\frac{1}{\sqrt{2}}\right)^{n-1} n \sum_{k=0}^{n} \binom{n-1}{k} a^{*k} a^{n-1-k}$$

$$= n :q^{n-1}:$$

$$\langle 1, :q^n: 1\rangle = \left(\frac{1}{\sqrt{2}}\right)^n \sum_{k=0}^{n} \binom{n}{k} \langle 1, a^{*k} a^{n-k} 1\rangle$$

Since $a1 = 0$, all terms but the $k=n$ term are zero. For this term we get $\langle 1, a^{*n} 1\rangle = \langle 1, z^n\rangle = 0$ for $n > 0$. So:

$$\langle 1, :q^n: 1\rangle = 0 \qquad \forall n > 0$$

as we wished to show.

Finally

$$:q^0: = \left(\frac{1}{\sqrt{2}}\right)^0 \binom{0}{0} a^{*0} a^0 = 1$$

so $:q^n: = \left(\frac{1}{\sqrt{2}}\right)^n \sum_{k=0}^{n} \binom{n}{k} a^{*k} a^{n-k}$ $n \in \mathbb{N}$ are the unique Wick powers of q. ∎

4) There exists a sequence $P_0, P_1, \ldots$ of polynomials satisfying

(i) $P_n'(q) = nP_{n-1}(q)$ $\quad n > 0$.

(ii) $\langle 1, P_n(q) 1 \rangle = 0$

(iii) $P_0(q) = 1$

Proof: Define $P_0(q) = 1$, $P_1(q) = q$. Then (i) clearly holds and $q1 = \frac{a+a^*}{\sqrt{2}} 1 = \frac{z}{\sqrt{2}} \Rightarrow \langle 1, P_1(q) 1 \rangle = \langle 1, \frac{z}{\sqrt{2}} \rangle = 0$ so (ii) holds as well. Now suppose $P_0, P_1, \ldots, P_{n-1}$ are defined. Define $P_n(q)$ by:

$$P_n(q) := n \int_0^q P_{n-1}(q_1)\, dq_1 - \left\langle 1, \int_0^q P_{n-1}(q_1) dq_1 \, 1 \right\rangle$$

where we write $\int_0^q - \, dq_1$ for the antiderivative with zero constant term. $P_n(q)$ satisfies (i) by definition of the antiderivative, and for (ii) we get

$$\begin{aligned}
\langle 1, P_n(q) 1 \rangle &= \left\langle 1, \left(n \int_0^q P_{n-1}(q_1) dq_1 - \left\langle 1, \int_0^q P_{n-1}(q_1) dq_1 \, 1 \right\rangle \right) 1 \right\rangle \\
&= \left\langle 1, n \int_0^q P_{n-1}(q_1) dq_1 \, 1 \right\rangle - \left\langle 1, \int_0^q P_{n-1}(q_1) dq_1 \, 1 \right\rangle \langle 1, 1 \rangle \\
&= 0
\end{aligned}$$

so by induction, the P_i exist. ∎