

5 April 2005

## Connections on Principal Bundles

We defined a principal  $G$ -bundle for any Lie group  $G$  to be a locally trivializable bundle of  $G$ -torsors. I.e. a smooth map

$$\begin{array}{ccc} P & \xrightarrow{\quad} & \text{total space} \\ p \downarrow & \xrightarrow{\quad} & \text{projection} \\ M & \xrightarrow{\quad} & \text{base space} \end{array}$$

where  $G$  acts on  $P$  (on the left for us; on the right for everyone else) in such a way that each fiber

$$P_x = p^{-1}(x) \quad (x \in M)$$

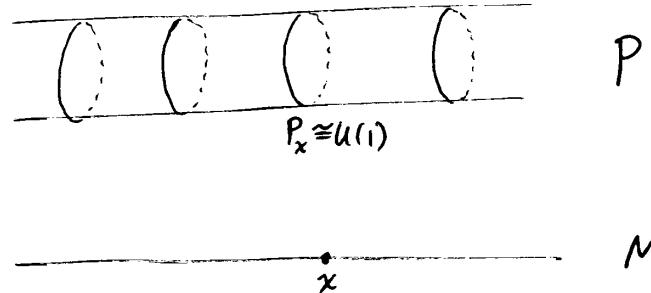
becomes a  $G$ -torsor (i.e. a manifold on which  $G$  acts which is isomorphic to  $G$  itself), and the whole setup is locally trivizilizable. This means that for any  $x \in M$  there's an open neighborhood  $U \ni x$  s.t.

$$\begin{array}{ccc} p^{-1}U & \xrightarrow[t]{\sim} & U \times G \\ & \searrow_{p|_{p^{-1}U}} \swarrow & \\ & U & \end{array}$$

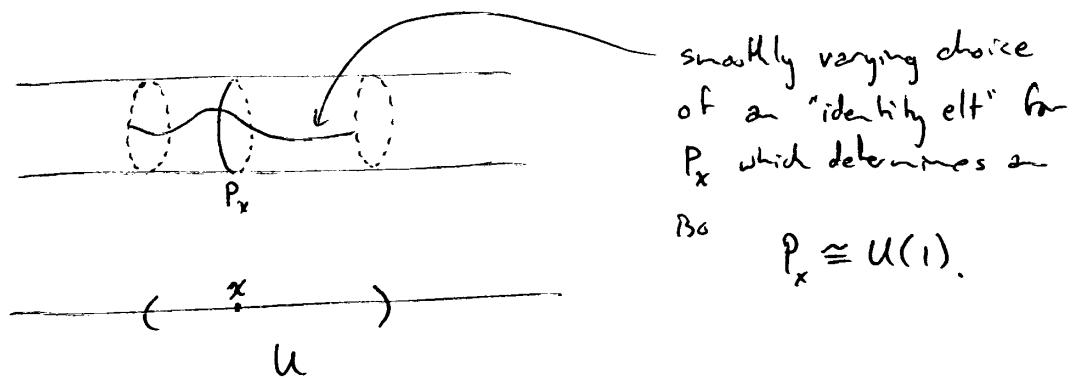
there exists a smooth map  $t$  with smooth inverse s.t. the diagram commutes &  $t$  preserves the action of  $G$ .

$$t(gy) = g t(y) \quad \forall y \in p^{-1}(U).$$

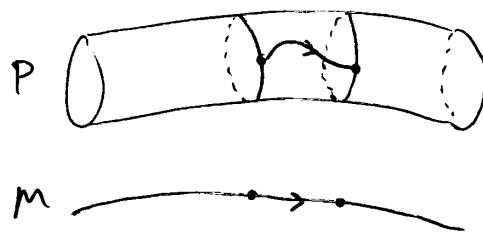
For example if  $G = U(1)$  you should visualize a principal  $G$ -bundle looks like this:



where each fiber is isomorphic as a space on which  $U(1)$  acts to  $U(1)$  itself, but not in a chosen way. A local trivialization looks like:

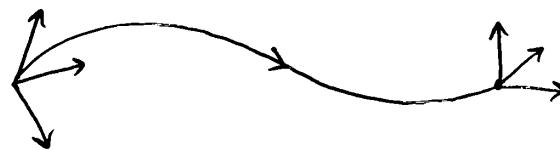


A "connection" is a rule for "moving points in  $P$  along curves in  $M$ ".

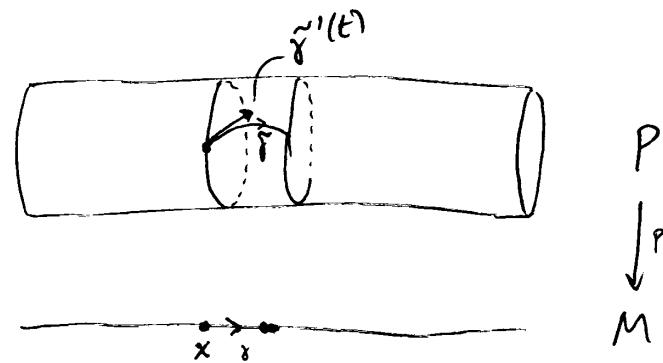


or "lifting" curves in  $M$  to curves in  $P$ .

For example, in general relativity  $G = GL(n)$ ,  $P$  is the frame bundle of  $M$ , & we want to "parallel transport" a frame along a curve in  $M$ :

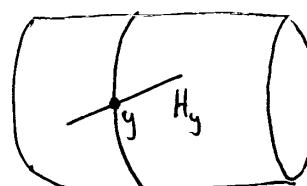


For this we should decide which tangent vectors to  $P$  count as "horizontal":



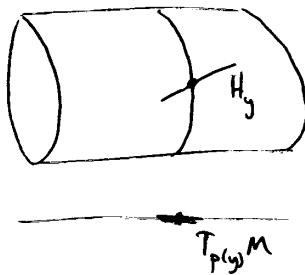
I.e. given a curve  $\gamma: [0, 1] \rightarrow M$  in the base space we want to lift it to a curve  $\tilde{\gamma}: [0, 1] \rightarrow P$  (i.e.  $p(\tilde{\gamma}(t)) = \gamma(t)$ ) in such a way that the tangent vector  $\tilde{\gamma}'(t)$  is always "horizontal".

So a connection will be a well-behaved choice of "horizontal space"  $H_y \subseteq T_y P$  for all  $y \in P$ :



Def - A connection  $H$  on the principal  $G$ -bundle  $P \xrightarrow{\rho} M$   
 $\Leftrightarrow$  choice of horizontal subspace  $H_y \subseteq T_y P$  for  $y \in P$   
 s.t.

- 1)  $H_y$  varies smoothly with  $y$ .
- 2)  $d\rho: T_y P \rightarrow T_{\rho(y)} M$  restricts to an isomorphism  
 $\sim d\rho: H_y \xrightarrow{\sim} T_{\rho(y)} M$



- 3) Given  $g \in G$  we have a map  $g: P \rightarrow P$   
 $y \mapsto gy$

so we get

$$dg: T_y P \rightarrow T_{gy} P$$

and we require

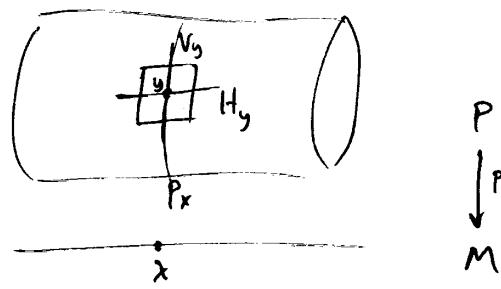
$$dg: H_y \rightarrow H_{gy} .$$

This is called equivariance.

There are lots of choices of "horizontal subspaces"  $H_y \subseteq T_y P$   
 satisfying  $H_y \subseteq T_y P$  satisfying 1)-3), but there's one choice  
 of vertical subspace  $V_y \subseteq T_y P$ , namely

$$V_y = \ker d\rho$$

In fact :



we have  $T_y P \cong H_y \oplus V_y$  precisely because of condition 2) :

$$0 \longrightarrow V_y \hookrightarrow T_y P \xrightarrow{dp} T_{p_y} M \longrightarrow 0 \quad (\text{splits})$$

$T_y P$  is the direct sum of  $\ker dp$  & any subspace (e.g.  $H_y$ !) on which  $dp$  is 1-1. Moreover

$$V_y \cong T_y P_x \subseteq T_y P$$

- vertical vectors are precisely those tangent to the fiber  $P_x$ .

To get a definition of connection more suitable for calculations, let's note that there's a God-given isomorphism

$$\alpha: V_y \xrightarrow{\sim} \mathfrak{o}_y$$

where  $\mathfrak{o}_y = T_y G$  is the Lie algebra of  $G$ . This is

plausible since  $V_y \cong T_y P_x$  & the fiber  $P_x$  is  $\cong$  to  $G$ ,  
but we get a specific isomorphism  $\alpha$  as follows:

$$f: P_x \rightarrow G$$

$z \mapsto g \in G$  s.t.  $gy = z$  ( $\exists! g$  since  $P_x$  is  
a  $G$ -torsor)

gives

Note:  $f$  is an iso  
of  $G$ -spzres

$$df: T_y P_x \xrightarrow{\sim} T_z G = \mathcal{O}_y$$

and this is our  $\alpha$ . So: we have a ~~good-given~~  
map

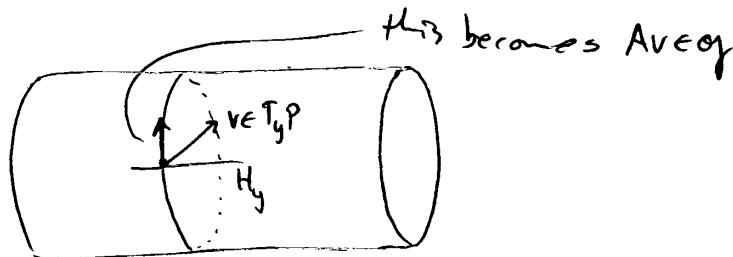
$$\alpha: V_y \rightarrow \mathcal{O}_y$$

but a connection will extend this to give a linear map

$$A: T_y P \rightarrow \mathcal{O}_y$$

for each  $y \in P$ . This goes as follows:

$$T_y P \xrightarrow{\sim} H_y \oplus V_y \xrightarrow{\text{projn onto } V_y} V_y \xrightarrow{\alpha} \mathcal{O}_y$$



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We've seen:

Def - A connection on a principal  $G$ -bundle  $p: P \rightarrow M$  is a choice of horizontal subspace  $H_y \subseteq T_y P$  for each  $y \in P$  such that

- 1)  $H_y$  varies smoothly with  $y$
- 2)  $dp: T_y P \rightarrow T_{py} M$  restricts to an isomorphism

$$dp: H_y \xrightarrow{\sim} T_{py} M$$

or equivalently:

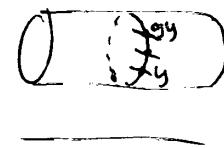
$$T_y P = V_y \oplus H_y$$

where  $V_y = \ker dp$ .

- 3) Equivariance:  $\forall g \in G$

$$dg: T_y P \rightarrow T_{gy} P$$

maps  $H_y$  to  $H_{gy}$



Given this definition, we can define "parallel transport":

Def - Given a connection  $H$  on  $p: P \rightarrow M$  & given a curve  $\gamma: [A, B] \rightarrow M$ , we say  $\tilde{\gamma}: [A, B] \rightarrow P$  is a horizontal lift of  $\gamma$  if:

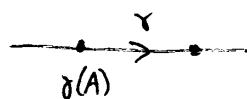
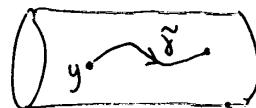
- 1)  $p \tilde{\gamma}(t) = \gamma(t)$  ("lift")
- 2)  $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}$  ("horizontal")

Given a horizontal lift  $\tilde{y}$  with  $\tilde{y}(0) = y$  we say that  $\tilde{y}(t)$  is the result of parallel transporting  $y$  along  $\gamma(t)$ .

For example, if  $P$  is a frame bundle: we can parallel transport  $y \in FM$  along a curve  $\gamma$  in  $M$  starting at  $py \in M$ .



Thm - Given a connection  $H$  on  $p: P \rightarrow M$ , a curve  $\gamma: [A, B] \rightarrow M$ , and a point  $y \in P$  with  $py = \gamma(A)$ , there exists a unique horizontal lift  $\tilde{\gamma}: [A, B] \rightarrow P$  with  $\tilde{\gamma}(A) = y$ .



Sketch of Proof: Solve this ODE:

$$\gamma'(t) = (dp)^{-1} \tilde{\gamma}'(t)$$

(where  $(dp)^{-1}$  is defined as the inverse of

$$dp: H_y \xrightarrow{\sim} T_{py} M$$

with this initial condition

$$\tilde{\gamma}(A) = y$$

Use the standard result on existence & uniqueness of solns of 1st ord. ODE.

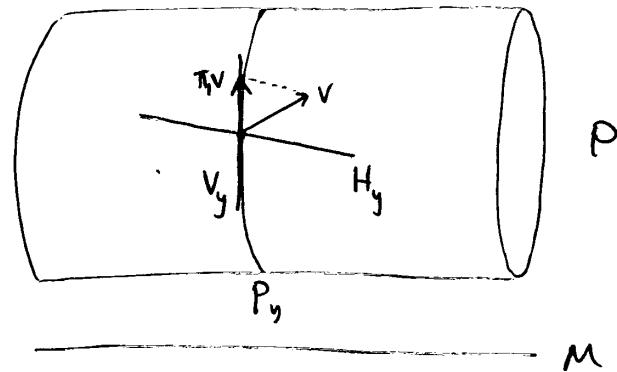
We also have begun to see another view of connections, which leads us to this theorem:

Thm — Suppose  $H$  is a connection on the principal  $G$ -bundle  $p: P \rightarrow M$ . Then for any  $y \in P$  we get a linear map

$$A_y : T_y P \rightarrow \mathfrak{g}$$

given by

$$T_y P \xrightarrow{\sim} V_y \oplus H_y \xrightarrow{\pi_1} V_y \xrightarrow{\sim} \mathfrak{g}$$



with  $\alpha$  the God-given isomorphism any principal bundle has.

These maps  $A_y$  ( $y \in P$ ) satisfy 3 properties

1)  $A_y$  varies smoothly with  $y$

2)  $A_y$  equals  $\alpha$  on  $V_y$ .

3)  $A_{gy}(dg(v)) = \text{Ad}(g) A_y(v) \quad \forall g \in G, v \in T_y P$

(where  $\text{Ad}(g): \mathfrak{g} \rightarrow \mathfrak{g}$  is the adjoint representation of  $G$  on  $\mathfrak{g}$ , given by differentiating the map

$$\begin{aligned} \text{AD}(g): G &\rightarrow G \\ x &\mapsto gxg^{-1} \end{aligned}$$

at  $1 \in G$ ) Conversely, any choice of linear map

$A_y: T_y P \rightarrow \mathfrak{g}$  satisfying 1)–3) comes from a unique connection  $H$  in this way.

In fact

$$H_y = \ker A_y.$$

Proof - See Choquet-Bruhat, DeWitt-Morette, & Dillard-Bleick's Analysis, Manifolds & Physics.  $\square$

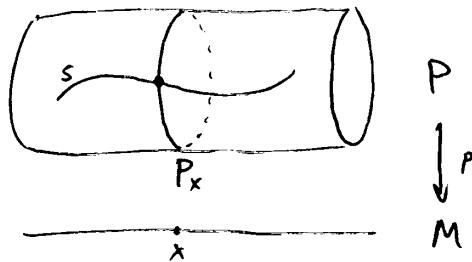
For short we write

$$A \in \Omega^1(P, \mathfrak{g}_f)$$

i.e.  $A$  is a  $\mathfrak{g}_f$ -valued 1-form on  $P$ , i.e. a smoothly varying family of linear maps

$$A_y : T_y P \rightarrow \mathfrak{g}_f.$$

We can think our connection as a  $\mathfrak{g}_f$ -valued 1-form  $A$  on  $M$  if we choose a section of  $p: P \rightarrow M$ , i.e. a map  $s: M \rightarrow P$  s.t.  $p \circ s = x$ :



We define  $A$  by

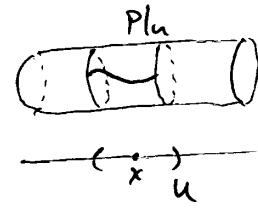
$$\underline{A}(v) = A_{s(x)}(ds(v))$$

for any  $v \in T_x M$ . Alas, we cannot usually find a "global"

section  $s: M \rightarrow P$ , but we can always find for any  $x \in M$   
 $\exists$  "local" section defined on some nbhd  $U \ni x$ , i.e.

$$s: U \rightarrow P|_U \subseteq P$$

s.t.  $p \circ s = 1$  on  $U$



In fact, we'll see:

- 1) A principal bundle  $P \rightarrow M$  has a section iff its trivializable:

$$\begin{array}{ccc} P & \xrightarrow{\sim} & M \times G \\ & \searrow p & \swarrow \\ & M & \end{array}$$

Since every principal bundle is locally trivializable,  
they all admit local sections.

- 2) When we construct  $\underline{A}$  from  $A$  this way, we can reconstruct  $A$  from  $\underline{A}$ , so we can think of  $A$  as "being"  $\underline{A}$ .