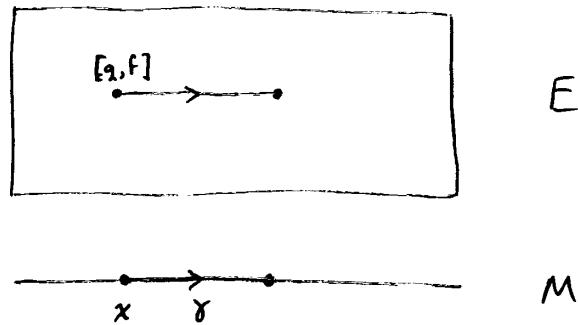


## Parallel Transport & Covariant Derivatives

Suppose  $p: P \rightarrow M$  is a principal  $G$ -bundle,  $F$  is a  $G$ -space,  $E = P \times_G F$  is the associated bundle, and  $A$  is a connection on  $P$ . Given a point in  $E$ , how can we parallel transport it along a curve  $\gamma$  in  $M$ ?



More precisely, given an equivalence class of pairs  $[q, f] \in P \times_G F$  with  $p(q) = x$  & a curve

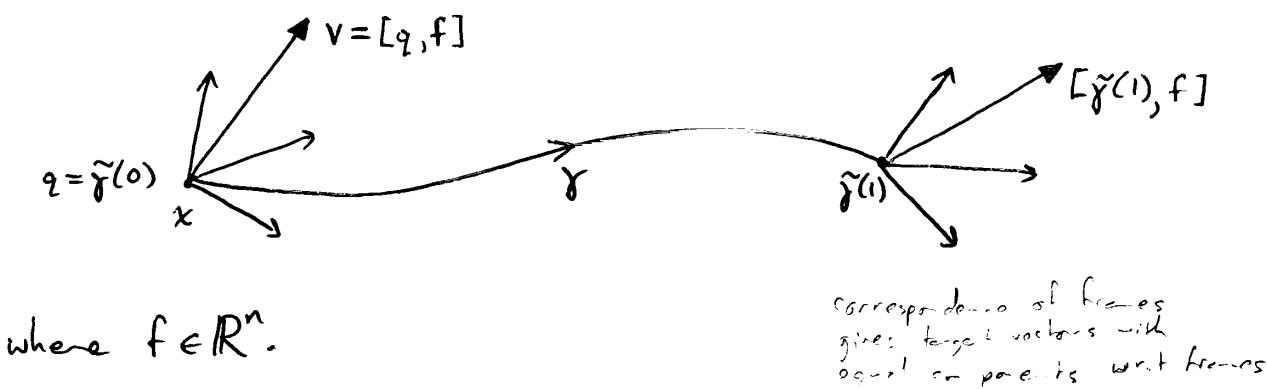
$$\gamma: [0, 1] \rightarrow M$$

with  $\gamma(0) = x$ , we want a nice "horizontal" way to lift  $\gamma$  to a curve in  $E$  starting at  $[q, f]$ . There's only one way to do this; namely: let  $\tilde{\gamma}: [0, 1] \rightarrow P$  be the horizontal lift of  $\gamma$  with  $\tilde{\gamma}(0) = q$ ; use this  $\tilde{\gamma}$  to drag  $q$  along, & don't mess with  $f$  at all: get a curve

$$t \mapsto [\tilde{\gamma}(t), f] \in E$$

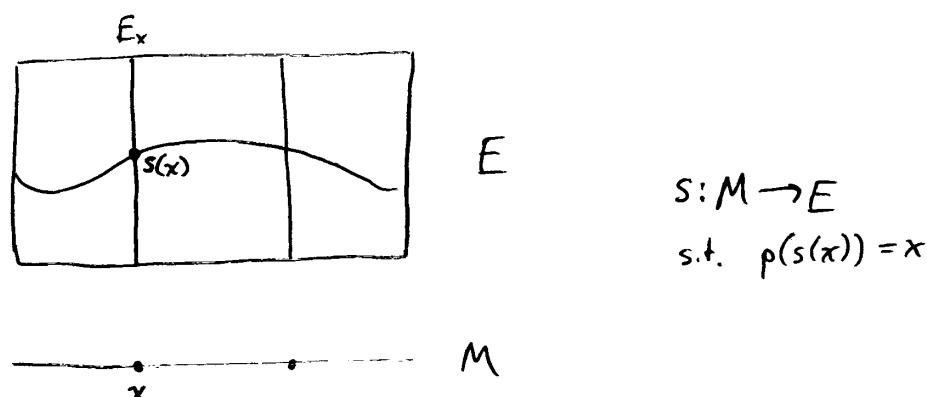
which lifts  $\gamma$ .

To see what this is like, let  $P$  be the frame bundle of  $M$  ( $G = G(n)$ ) &  $F = \mathbb{R}^n$  so that  $E = TM$ . A connection on  $P$  tells us how to parallel transport frames; we then use this to parallel transport tangent vectors:



In general we call  $[\tilde{\gamma}(t), f]$  the result of parallel transporting  $[q, f]$  along  $\gamma$ .

Now suppose  $F$  is a vector space &  $G$  acts linearly on  $F$ , so  $E = P \times_G F$  is a vector bundle. Now we can talk about the "covariant derivative" of a section  $s$  of  $E$ :



Given  $v \in T_x M$ , let's figure out the covariant derivative of  $s$  in the  $v$  direction,  $D_v s$ . To do this let's pick

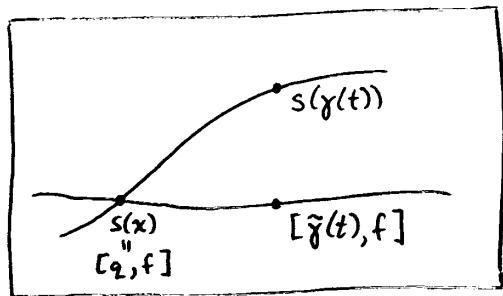
a curve  $\gamma: [0,1] \rightarrow M$  with  $\gamma(0) = x$  and  $\gamma'(0) = v$

We want to see how  $s(\gamma(t))$  changes as  $t$  changes.

The obvious formula

$$\lim_{t \rightarrow 0} \frac{s(\gamma(t)) - s(x)}{t} \quad (\text{wrong!})$$

is wrong since  $s(\gamma(t))$  &  $s(x)$  are in different vector spaces!  
Better:



let's compare  $s(\gamma(t))$  to the result of parallel transporting  $s(x)$  along  $\gamma$ , namely  $[\tilde{\gamma}(t), f]$  (if  $s(x) = [g, f]$ ).

We get

$$D_v s = \lim_{t \rightarrow 0} \frac{s(\gamma(t)) - [\tilde{\gamma}(t), f]}{t} \in E_x$$

or

$$D_v s = \left. \frac{d}{dt} s(\gamma(t)) - [\tilde{\gamma}(t), f] \right|_{t=0}$$

Covariant differentiation satisfies various rules: given any sections  $s, s'$  of  $E$  ( $s \in \Gamma(E)$  means  $s$  is a section of  $E$ ), any  $v \in T_x M$ , any smooth function  $f: M \rightarrow \mathbb{R}$  ( $f \in C^\infty(M)$ ) we have

$$\star \left\{ \begin{array}{l} D_v(s + s') = D_v s + D_v s' \\ D_v(cs) = c D_v s \quad c \in \mathbb{R} \\ D_v(fs) = v(f)s + f D_v s \end{array} \right.$$

(where  $v(f)$  denotes the directional derivative of  $f$  in the  $v$  direction)

Moreover if  $v, v' \in T_x M$

$$\star \star \left\{ \begin{array}{l} D_{v+v'}(s) = D_v s + D_{v'} s \\ D_{cv}(s) = c D_v s \quad c \in \mathbb{R} \end{array} \right.$$

Note:  $s$  is a section of  $E$   
 $f$  is a section of the trivial bundle  $M \times \mathbb{R}$   
but  $v$ , alas, is not a section of  $TM$

So now, let  $v$  be a section of  $TM$ , i.e. a vector field:  $\forall x \in M \quad v(x) \in T_x M$ . We can define  $D_v s$  for a vector field  $v$  by:

$$(D_v s)(x) = D_{v(x)} s$$

so that  $D_v s \in \Gamma(E)$  whose value at  $x \in M$  is  $D_{v(x)} s$ .

In this new improved setup,  $\star$  still hold (& middle rule is redundant: treat  $c$  as a constant function) &  $\star \star$  gets

enhanced to

$$D_{v+v'} s = D_v s + D_{v'} s$$

$$D_{fv} s = f D_v s$$

Two calculations:

- 1) Suppose we have 2 connections on  $P$  hence two covariant differentiation operators  $D$  &  $D'$ . Calculate:

$$\begin{aligned} (D_v - D'_v)(fs) &= D_v(fs) - D'_v(fs) \\ &= v(f)s + f D_v s - v(f)s - f D'_v s \\ &= f(D_v - D'_v)s \end{aligned}$$

so

$$(D_v - D'_v)f = f(D_v - D'_v)$$

so  $D_v - D'_v$  is linear over  $C^\infty(M)$  even though  $D_v, D'_v$  are not. Moral: differences of  $D_v$ 's are very nice and simple. (Torsions!)

- 2) Given vector fields  $v, w$  on  $M$  ( $v, w \in \text{Vect}(M)$ ),  
 $s \in \Gamma(E)$ ,  $f \in C^\infty(M)$

$$\begin{aligned} [D_v, D_w](fs) &= D_v D_w(fs) - D_w D_v(fs) \\ &= D_v(w(f)s + f D_w s) - D_w(v(f)s + f D_v s) \\ &= v(w(f))s + w(f)D_v s + v(f)D_w s + f D_v D_w s \\ &\quad - w(v(f))s - v(f)D_w s - w(f)D_v s - f D_w D_v s \\ &= v(w(f))s - w(v(f))s + f[D_v, D_w]s \end{aligned}$$

$$= ([v, w] f) s + f [D_v, D_w] s$$

In fact  $[v, w]$  is a vector field, the Lie bracket of  $v \& w$ .  
 The cool way to summarize what we've done is

$$( [D_v, D_w] - D_{[v, w]} ) f = f ( [D_v, D_w] - D_{[v, w]} )$$

Proof: since

$$D_{[v, w]} = ([v, w] f) s + f D_{[v, w]} s$$

we get

$$\begin{aligned} & ( [D_v, D_w] - D_{[v, w]} ) f \\ &= f [D_v, D_w] + [v, w] f - [v, w] f - f D_{[v, w]} \\ &= f ( [D_v, D_w] - D_{[v, w]} ) \end{aligned}$$

Moral:  $[D_v, D_w] - D_{[v, w]}$  is important — it's the curvature of the connection. It's zero iff our connection is flat.

## Calculating Covariant Derivatives

Last time we defined the covariant derivative  $D_v s$  of a section  $s$  of a vector bundle  $E$  in the  $v$  direction, where  $v \in T_x M$ . Now let's show how to calculate this in terms of our description of a connection as a  $\mathfrak{g}$ -valued 1-form  $A$  on the principal bundle  $P \rightarrow M$  or as a  $\mathfrak{g}$ -valued 1-form  $A$  on  $M$  itself, given that  $P$  has been (locally) trivialized, i.e. we've chosen an isomorphism

$$P|_U \xrightarrow{\sim} U \times G$$

for some open set  $U \subseteq M$  containing  $x \in M$ .

Recall the cast of characters:

$G$  Lie Group

$P$   
 $\downarrow$   
 $M$  Principal  $G$ -bundle

$A$  connection on  $P$  with horizontal subspaces  $H_y \subseteq T_y P$

$F$  a vector space on which  $G$  acts linearly

$E = P \times_G F$  the associated vector bundle

Given all this, suppose

$s$  is a section of  $E$   
&  $v \in T_x M$

Then we defined

$$D_v s = \frac{d}{dt} s(\gamma(t)) - [\tilde{\gamma}(t), f] \Big|_{t=0}$$

where

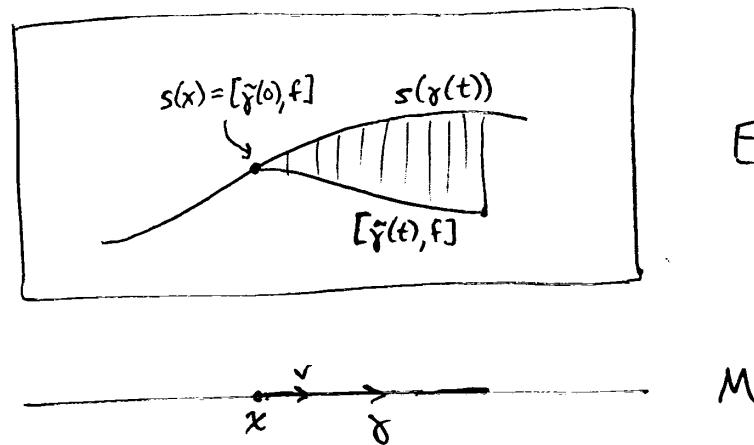
$$\gamma: [0, 1] \rightarrow M$$

is a curve in  $M$  with  $\gamma(0) = x$   $\gamma'(0) = v$ , and

$$\tilde{\gamma}: [0, 1] \rightarrow P$$

is the horizontal lift of  $\gamma$  with  $\tilde{\gamma}'(0) = f$

$$s(x) = [\tilde{\gamma}(0), f] \in P \times_a F = E$$



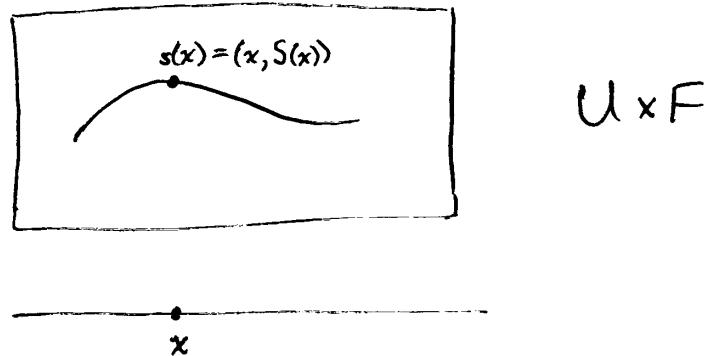
Now let's calculate  $D_v s$  in terms of  $v, s$  & the connection 1-form  $A$  (or  $\underline{A}$ ) by choosing a local trivialization of  $P$ . To do this, choose a neighborhood  $U \ni x$  & an isomorphism of principal  $G$ -bundles:

$$P|_U \xrightarrow{\sim} U \times G$$

This lets us treat  $P|_U$  as "being"  $U \times G$ , and

$$E|_U = P \times_a F|_U = P|_U \times_a F \cong U \times G \times_a F \cong U \times F$$

so we'll treat  $E|_U$  as "being"  $U \times F$ . This lets us think of our section  $s$  as just a function:



We have

$$s(x) = (x, S(x)) \in U \times F$$

for some function

$$S: U \longrightarrow F$$

This lets us write

$$s(\gamma(t)) = (\gamma(t), S(\gamma(t))) \in U \times F$$

It also lets us write  $[\tilde{\gamma}(t), f] \in P_U^G F$  as an actual pair in  $U \times F$ , once we figure out what  $\tilde{\gamma}(t)$  looks like as an element of  $U \times G$ . It's:

$$\tilde{\gamma}(t) = (\gamma(t), g(t)) \in U \times G$$

Note: first component must be  $\gamma(t)$  since  $\tilde{\gamma}(t)$  is a lift of  $\gamma(t)$ . The second component is determined by the fact that  $\tilde{\gamma}(t)$  is horizontal — so the connection

$A$  must be involved in defining  $g(t)$ , but let's not figure this out yet.

Anyway, we have:

$$[\tilde{\gamma}(t), f] \in P|_u \times_G F$$

||

$$[(\gamma(t), g(t)), f]$$

||

$$[(\gamma(t), 1)g(t), f]$$

||

$$[(\gamma(t), 1), g(t)f]$$

$\downarrow$

$$(\gamma(t), g(t)f) \in U \times F$$

compare:

$$E|_u = P \times_G F|_u$$

$$= P|_u \times_G F$$

$$\cong U \times G \times_G F$$

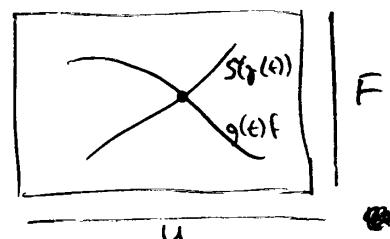
$$\cong U \times F$$

So we see that using our local trivialization, we have

$$\begin{aligned} D_v s &= \frac{d}{dt} [s(\gamma(t)) - [\tilde{\gamma}(t), f]] \Big|_{t=0} \\ &= \frac{d}{dt} [(\gamma(t), s(\gamma(t))) - (\gamma(t), g(t)f)] \Big|_{t=0} \\ &= \frac{d}{dt} s(\gamma(t)) - g(t)f \Big|_{t=0} \in F. \end{aligned}$$

Note

$$s(\gamma(t)) - g(t)f = 0 \text{ when } t=0:$$



which lets us express  $f$  as:

$$f = g(0)^{-1} S(\gamma(0))$$

In fact, wlog we can assume  $g(0) = 1$  since

$$[(\gamma(0), g(0)), f] = [(\gamma(0), 1), f'] \text{ where } f' = g(0)^{-1} f.$$

Then

$$f = S(\gamma(0))$$

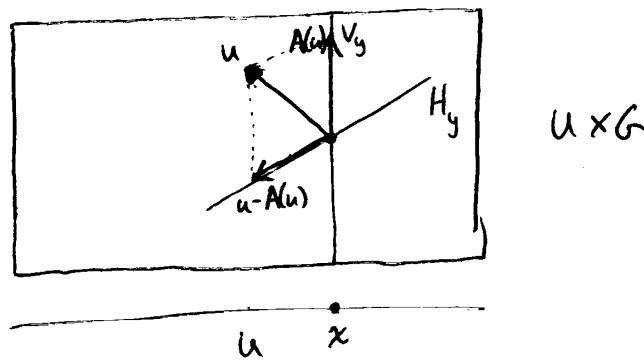
So

$$\begin{aligned} D_v s &= \frac{d}{dt} S(\gamma(t)) - g(t) S(\gamma(0)) \Big|_{t=0} \\ &= dS(\gamma'(0)) - g'(0) S(\gamma(0)) \\ &= dS(v) - g'(0) S(x) \end{aligned}$$

Now, we claim that

$$g'(0) = -\underline{A}(v)$$

where  $\underline{A} \in \Omega^1(U, g)$  is the connection.



Our connection 1-form  $A \in \Omega^1(U \times G, \text{of})$  does this

$$T_y P \cong V_y \oplus H_y \xrightarrow{\pi} V_y \cong \text{of}$$

$\curvearrowright_A$

$\tilde{\gamma}'(t)$  is horizontal, so  $\tilde{\gamma}'(0) \in H_y$  so

$$A(\tilde{\gamma}'(0)) = 0$$

But

$$\tilde{\gamma}(t) = (\gamma(t), g(t)) \in U \times G$$

so

$$\tilde{\gamma}'(0) = (\gamma'(0), g'(0)) \in T_x U \times \text{of}$$

so

$$\begin{aligned} 0 &= A(\tilde{\gamma}'(0)) \\ &= A(\gamma'(0), g'(0)) \\ &= A(\gamma'(0), 0) + A(0, g'(0)) \\ &= A(v, 0) + g'(0) \quad \downarrow A \text{ is the identity} \\ &= \underline{A}(v) + g'(0) \quad \text{on vertical vectors.} \end{aligned}$$

so

$$g'(0) = -\underline{A}(v).$$

So, we've shown that in terms of a local trivialization,

$$D_v s = dS(v) + \underline{A}(v)s$$

or blurring the distinction between  $s$  &  $S$  ...

$$\begin{aligned} D_v s &= v(s) + \underline{A}(v)s \\ \text{or } D_v &= v + \underline{A}(v) \end{aligned}$$