

26 April 2005

Covariant Derivatives

Let $P \rightarrow M$ be a principal G -bundle with a connection A and let F be a representation of G , let $E = P \times_G F$ be the associated vector bundle, and choose a trivialization of P over $U \subseteq M$.

We've seen that the trivialization lets us write

$$P|_U = U \times G$$

$$E|_U = U \times F$$

and lets us write sections of $E|_U$, say $s \in \Gamma(E|_U)$, as functions $s: U \rightarrow F$. (Last time we called the function S , but now we'll relax a bit.) It also lets us write the connection as a g_F -valued 1-form on U , $A \in \Omega^1(U, g_F)$. (Last time we called this 1-form \underline{A} , but now we'll relax some more). The main result last time was to show that the covariant derivative $D_v s$ of $s \in \Gamma(E|_U)$ in the direction $v \in T_x U$ was given by

$$D_v s = v(s) + \underbrace{A(v)}_{\in g_F} \underbrace{s}_{\in F}$$

where g_F acts on F since G does. It easily follows that if $v \in \text{Vect}(U)$ we get $D_v s \in \Gamma(E|_U)$ with

$$D_v s = v(s) + A(v)s$$

If we choose coordinates x^i on U then we get a basis of vector fields, the coordinate vector fields

$$\partial_i := \frac{\partial}{\partial x^i}$$

and we define $D_i := D_{\partial_i}$ & $A_i := A(\partial_i) \in C^\infty(U, \mathfrak{g})$.

Then we get

$$D_i s = \partial_i s + A_i s$$

$$\text{or } D_i = \partial_i + A_i$$

for short.

We know partial derivatives commute:

$$[\partial_i, \partial_j] = 0$$

but what about these covariant partial derivatives D_i :

$$[D_i, D_j] = 0 ?$$

Let's see:

$$\begin{aligned} [D_i, D_j] s &= D_i D_j s - D_j D_i s \\ &= (\partial_i + A_i)(\partial_j + A_j)s - (\partial_j + A_j)(\partial_i + A_i)s \\ &= \partial_i(A_j s) - A_j(\partial_i s) - (\partial_j(A_i s) - A_i(\partial_j s)) + [A_i, A_j]s \\ &= (\partial_i A_j - \partial_j A_i + [A_i, A_j])s \end{aligned}$$

using the product rule, $[\partial_i, \partial_j] = 0$, & the fact that F is a representation of \mathfrak{g} .

When Yang & Mills did this calculation, they noticed that when \mathfrak{g} was abelian, e.g. $\mathfrak{g} = u(1)$, this reduced to

$\partial_i A_j - \partial_j A_i$, which is the electromagnetic field F_{ij} corresponding to the vector potential A , so we can use it to generalize electromagnetism! They generalized Maxwell's equations & got the "Yang-Mills equations".

So we don't have $[D_i, D_j] = 0$, but $[D_i, D_j]$ is an interesting quantity. Earlier we considered this:

$$F(v, w) = [D_v, D_w] - D_{[v, w]}$$

& we saw that it's $C^\infty(M)$ -linear:

$$F(v, w)(fs) = f F(v, w)s \quad \forall s \in \Gamma(E) \\ f \in C^\infty(M)$$

In the special case where $v = \partial_i$, $w = \partial_j$ are coordinate vector fields, $[\partial_i, \partial_j] = 0$ so

$$F(\partial_i, \partial_j) = [D_i, D_j]$$

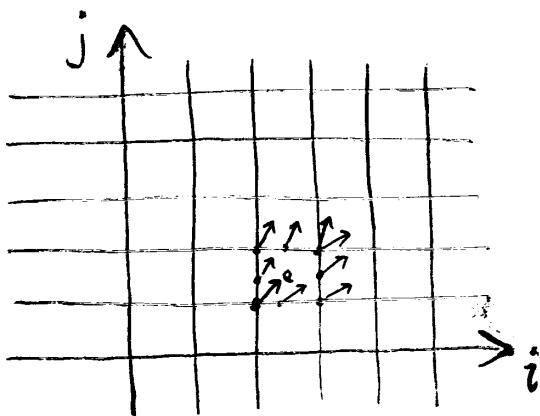
& if we are then able to pick a trivialization of P we get

$$F(\partial_i, \partial_j) = \partial_i A_j - \partial_j A_i + [A_i, A_j],$$

and we write

$$F_{ij} := F(\partial_i, \partial_j).$$

F is called the curvature of A and it's related to parallel transport around a small rectangle:



If we take $e \in E_x$ & parallel transport by ε in the i th direction, then ε in the j th direction, we don't get the same result as if we did it in the other order! The difference between these two is some vector $e_{ij}(\varepsilon)$, and

$$\lim_{\varepsilon \rightarrow 0} \frac{e_{ij}(\varepsilon)}{\varepsilon^2} = \underbrace{F_{ij}}_{\text{of } F} e$$

We can generalize all of calculus with covariant partials D_i replacing ordinary partials ∂_i . For example, in ordinary calculus we have differential forms, e.g. p -forms

$$\Omega^p M = \left\{ \omega : \omega(x) : T_x M \times \dots \times T_x M \rightarrow \mathbb{R} \text{ is multilinear \& antisymmetric and depends smoothly on } x \in M \right\}$$

& an exterior derivative operator

$$d : \Omega^p M \rightarrow \Omega^{p+1} M$$

given in local coordinates by

$$\begin{aligned} \omega &= \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \\ d\omega &= \underset{\Downarrow}{\partial_{i_0}} \omega_{i_1 \dots i_p} dx^{i_0} \wedge \dots \wedge dx^{i_p} \end{aligned}$$

where $dx^{i_1} \wedge \dots \wedge dx^{i_p}$ is the differential form that gives 1 when we apply it to $(\partial_{i_1}, \dots, \partial_{i_p}) \in T_x M \times \dots \times T_x M$ &

0 when we apply it to any list of coordinate vector fields or if i_1, \dots, i_p are not all distinct.
that's not a permutation of this one! Note we have $d^2 = 0$

i.e.

$$\Omega^p M \xrightarrow{d} \Omega^{p+1} M \xrightarrow{d} \Omega^{p+2} M$$

\circ

because $[\partial_i, \partial_j] = 0$:

$$\omega = \omega_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

\Downarrow

$$d\omega = \partial_j \omega_{i_1, \dots, i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}$$

\Downarrow

$$d^2\omega = \partial_j \partial_i \omega_{i_1, \dots, i_p} dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_p}$$

$$= 0 \quad \text{since } \partial_j \partial_i = \partial_i \partial_j \text{ but } dx^i \wedge dx^j = -dx^j \wedge dx^i$$

Next time we'll define the covariant exterior derivative d_A of E -valued p -forms & see that $d_A^2 \neq 0$ because $[\partial_i, \partial_j] \neq 0$.

28 April 2005

Covariant Exterior Derivatives

Suppose $P \rightarrow M$ is a principal G -bundle, V is a vector space on which G has a representation, $E = P \times_G V$ the associated vector bundle, & A a connection on P . Let

$$\Omega^p(M, E) = \left\{ \omega : \omega(x) : \overbrace{T_x M \times \dots \times T_x M}^p \rightarrow E_x \text{ such that } \omega(x) \text{ is multilinear, antisymmetric & varies smoothly with } x \in M \right\}$$

be the vector space of E -valued p -forms.

We can define the covariant exterior derivative of such a thing using the connection A . This is an operator

$$d_A : \Omega^p(M, E) \rightarrow \Omega^{p+1}(M, E).$$

Using local coordinates on $U \subseteq M$ & a trivialization of $P|_U \cong U \times G$ (hence $E|_U \cong U \times V$), we can write a formula for d_A which looks almost like our previous formula for d .

Namely:

$$\omega = \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$



$$d_A \omega = D_{i_0} \omega_{i_1 \dots i_p} dx^{i_0} \wedge \dots \wedge dx^{i_p}$$

where $\omega_{i_1 \dots i_p}$ is a section of $E|_U$, regarded as a V -valued function on U & D_i is the covariant partial derivative in the i th coordinate direction:

$$D_i = \partial_i + A_i$$

where A_i is a g -valued function on U . So

$$D_{i_0} \omega_{i_1 \dots i_p} = \partial_{i_0} \omega_{i_1 \dots i_p} + \underbrace{A_{i_0} \omega_{i_1 \dots i_p}}_{\text{defined using rep of } g \text{ on } V.}$$

We should now check that $d_A \omega$ is independent of the choice of coordinates & trivialization ... but we won't.

Next: is $d_A^2 = 0$?

$$\omega = \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$



$$d_A \omega = D_j \omega_{i_1 \dots i_p} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}$$



$$d_A^2 \omega = D_i D_j \omega_{i_1 \dots i_p} dx^i \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

$$= \frac{1}{2} [D_i, D_j] \omega_{i_1 \dots i_p} dx^i \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

$$\begin{aligned} \text{since } D_i D_j s \, dx^i \wedge dx^j &= \frac{1}{2} (D_i D_j s \, dx^i \wedge dx^j + D_j D_i s \, dx^j \wedge dx^i) \\ &= \frac{1}{2} (D_i D_j s \, dx^i \wedge dx^j - D_j D_i s \, dx^i \wedge dx^j) \\ &= \frac{1}{2} [D_i, D_j] s \, dx^i \wedge dx^j \end{aligned}$$

Moreover:

$$\begin{aligned} [D_i, D_j] s &= \underbrace{F(\partial_i, \partial_j)}_{\substack{\text{op-valued fn.} \\ /}} s \\ &= F_{ij} s \quad \text{where } F \text{ is the curvature of } A. \end{aligned}$$

So

$$\begin{aligned} d_A^2 \omega &= \left(\frac{1}{2} F_{ij} dx^i \wedge dx^j \right) \wedge \omega \\ &\quad \text{is op-valued 2-form} \\ &\quad \text{... given by } Ad(E) \text{-val 2-form} \\ &= F \wedge \omega \end{aligned}$$

where now

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$$

is a op-valued 2-form — another way of thinking about the curvature.

Note this is consistent with the notation $F(\alpha_i, \alpha_j)$, since here we are feeding 2 vector fields into our α_j -valued 2-form and getting a α_j -val. function (also called F_{ij}).

Also: in the expression $F \wedge \omega$ we're using this map

$$\wedge : \Omega^q(U, \alpha_j) \times \Omega^p(U, V) \longrightarrow \Omega^{q+p}(U, V)$$

$$(\alpha dx^{i_1} \wedge \dots \wedge dx^{i_q}, v dx^{j_1} \wedge \dots \wedge dx^{j_p}) \mapsto \alpha(v) dx^{i_1} \wedge \dots \wedge dx^{i_q} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

So in short:

$$d_A^2 = F \wedge$$

and in particular $d_A^2 = 0$ when the connection A is flat, meaning $F = 0$.

Alas, all these calculations were local, using coordinates on U & trivialization of $P|_U$. Question: what is F really, before we make these choices? It's not really a α_j -valued 2-form until you choose a trivialization of $P|_U$. F is really an $Ad(P)$ -valued 2-form!

What's flat? $Ad(P)$ is a vector bundle associated to P in a very natural way. We get an associated vector bundle $E = P \times_G V$ whenever we have a vector space V on which G acts linearly. It's natural to use $V = \alpha_j$ with the adjoint representation

$$Ad(g)v = gvg^{-1}$$

(This formula is only good when G , hence \mathfrak{g}_f , is a metric Lie gp. — but this covers all the cases we care about.) This gives a vector bundle called $\text{Ad}(P)$:

$$\text{Ad}(P) = P \times_G \mathfrak{g}_f$$

In physics and geometry, almost always when you might naively think you have a \mathfrak{g}_f -valued p -form, it's really an $\text{Ad}(P)$ -valued p -form. For example:

i) We saw that if A & A' are connections on P , the covariant derivatives D & D' satisfy

$$(D_v - D'_v)f s = f(D_v - D'_v)s$$

for all $v \in \text{Vect}(M)$, $f \in C^\infty(M)$ & $s \in \Gamma(E)$. In fact $D_v - D'_v$ is best viewed as an $\text{Ad}(P)$ -valued function, or 0-form. This depends linearly on $v \in \text{Vect}(M)$, so $D - D'$ is an $\text{Ad}(P)$ -valued 1-form.

If we pick a trivialization of P we can write:

$$D_v s = v s + \underbrace{A(v)s}_{\text{og-val fn.}}$$

$$D'_v s = v s + \underline{A}'(v)s$$

so

$$D_v - D'_v = \underline{A}(v) - \underline{A}'(v)$$

is a \mathfrak{g}_f -val function, &

$$D - D'$$

is a \mathfrak{g}_f -val 1-form. But this viewpoint requires a trivialization!

2) F , the curvature is really an $\text{Ad}(P)$ -valued
2-form.