

## Gauge Transformations

Gauge theory has interesting symmetries called gauge transformations. These are "purely mathematical" symmetries in the sense that they act to permute different mathematical descriptions of the same physical situation — akin to a change of coordinates. Let's describe gauge transformations and how they act on:

- 1) The principal bundle  $P \rightarrow M$  — gauge transformations are just symmetries of this!
- 2) The associated bundle  $E = P \times_G F$
- 3) Sections  $s: M \rightarrow E$  of the associated bundle. (These are called "fields" in physics.)
- 4) Connections on  $P$ , and their covariant derivative operators

$$D_v: \Gamma(E) \rightarrow \Gamma(E)$$

where  $v \in \text{Vect}(M)$  and  $E = P \times_G V$  is a vector bundle (where now  $F = V$  is a vector space on which  $G$  acts linearly)

(These connections are called "gauge fields" in physics)

- 5) The curvature of a connection.

First, given two principal  $G$ -bundles  $P, P'$  over  $M$ , an isomorphism  $\alpha: P \rightarrow P'$  is a diffeomorphism  $\alpha: P \rightarrow P'$  such that

$$\textcircled{1} \begin{array}{ccc} P & \longrightarrow & P' \\ p \downarrow & & \downarrow p' \\ & M & \end{array} \text{ commutes, where } p, p' \text{ are the projections down to } M$$

and

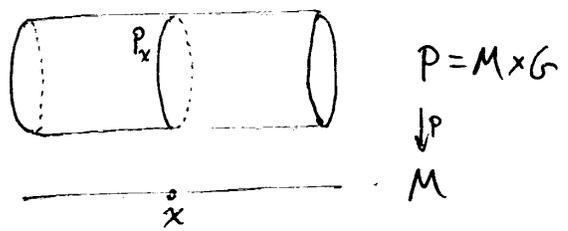
$$\textcircled{2} \begin{array}{ccc} P \times G & \xrightarrow{\alpha \times 1} & P' \times G \\ \downarrow & & \downarrow \\ P & \longrightarrow & P' \end{array}$$

commutes, where  $A, A'$  are the actions of  $G$  on  $P, P'$ , i.e.

$$\alpha(qg) = \alpha(q)g \quad (q \in P, g \in G).$$

1) A gauge transformation is then an automorphism  $\alpha: P \rightarrow P$  of the principal  $G$ -bundle  $P$ , i.e. an iso. from  $P$  to itself.

If  $P$  is the trivial  $G$ -bundle over  $M$  (i.e.  $P = M \times G$ ) then a gauge transformation  $\alpha: P \rightarrow P$  amounts to just a function  $g: M \rightarrow G$ .



In this picture  $\alpha$  must map the fiber  $P_x$  to itself (by ①) and it must commute with all "rotations" (by ②), where "rotations" is the correct word for  $G = U(1)$ . Why?

A map  $f: G \rightarrow G$  that commutes with all right multiplications:

$$f(hg) = f(h)g \quad \forall g, h \in G$$

must be left multiplication by some  $g' \in G$ :

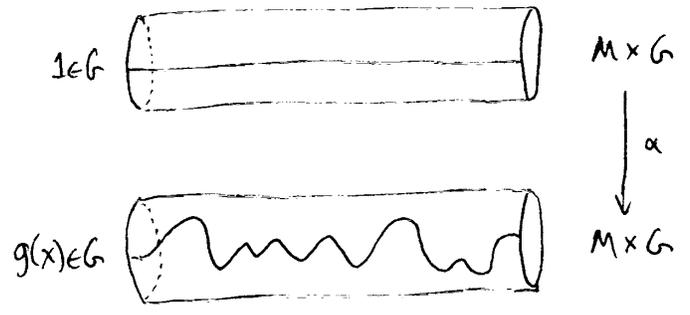
$$f(h) := g'h \quad \forall h \in G.$$

Why is this true? If  $f$  satisfies  $f(hg) = f(h)g$  define  $g' = f(1)$  and then note  $f(h) = f(1h) = f(1)h = g'h$ , as desired.

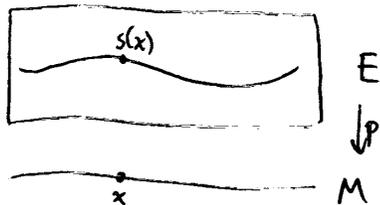
So: given a gauge transformation  $\alpha: P \rightarrow P$ , where  $P = M \times G$ , we have

$$\alpha(x, h) = (x, g(x)h)$$

for some function  $g: M \rightarrow G$ .



3) Given a gauge transformation  $\alpha: P \rightarrow P$ , it acts on  $\Gamma(E)$ , the set of sections of  $E$  as follows. A section  $s \in \Gamma(E)$  is a map  $s: M \rightarrow E$  s.t.  $p(s(x)) = x$ .



The gauge transformation  $\alpha$  acts on  $s$  to give a new section  $\alpha(s) \in \Gamma(E)$ :

$$\alpha(s)(x) = \tilde{\alpha}(s(x)).$$

If  $P$  (hence  $E$ ) is trivial, we can think of a section  $s \in \Gamma(E)$  as a function  $s: M \rightarrow F$ , and in this description we have

$$\alpha(s)(x) = g(x)s(x) \quad \forall x \in M$$

4) Next, suppose  $E = P \times_G V$  is a vector bundle. A connection  $A$  on  $P$  gives a covariant derivative operator

$$D_v: \Gamma(E) \rightarrow \Gamma(E) \quad \text{for each } v \in \text{Vect}(M).$$

Given a gauge transformation  $\alpha: P \rightarrow P$  we get a new covariant derivative operator  $D'$  by demanding that

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{D_v} & \Gamma(E) \\ \alpha \downarrow & & \downarrow \alpha \\ \Gamma(E) & \xrightarrow{D'_v} & \Gamma(E) \end{array} \quad \text{commute } \forall v, \text{ so } D'_v := \alpha D_v \alpha^{-1}.$$

Now if  $P$  (hence  $E$ ) is trivial, we can choose local coordinates and write

$$D_v s = v s + A(v) s$$

where  $A \in \Omega^1(M, \mathfrak{g})$  describes our connection. In particular,

$$D_i s = \partial_i s + A_i s.$$

This is only true for a trivial principal  $G$ -bundle, but of course every principal  $G$ -bundle is locally trivialisable, so we can always use this description of gauge transformations locally.

2) Next, given an associated bundle  $E = P \times_G F$ , any gauge transformation  $\alpha: P \rightarrow P$  gives an isomorphism

$$\tilde{\alpha}: E \rightarrow E$$

by

$$\tilde{\alpha}[q, f] = [\alpha(q), f] \quad \forall q \in P, f \in F$$

Recall that  $[q, f]$  is an equivalence class with

$$[qg, g^{-1}f] = [q, f]$$

so we need to check that  $\tilde{\alpha}$  is well defined:

$$\begin{aligned} \tilde{\alpha}[qg, g^{-1}f] &= [\alpha(qg), g^{-1}f] \\ &= [\alpha(q)g, g^{-1}f] \\ &= [\alpha(q), f] \\ &= \tilde{\alpha}[q, f]. \quad \checkmark \end{aligned}$$

If  $P$  is trivial ( $P = M \times G$ ), then  $E$  is trivial too:

$$E = (M \times G) \times_G F \cong M \times F$$

and in this description of  $E$  we have

$$\begin{aligned} \tilde{\alpha}: M \times F &\longrightarrow M \times F \\ (x, f) &\longmapsto (x, g(x)f) \end{aligned}$$

where we think of  $\alpha$  as the  $G$ -valued function

$$g: M \rightarrow G$$

We leave the proof to the reader.

Then:

$$\begin{aligned}
 (D'_i s)(x) &= g(x) D_i (g(x)^{-1} s)(x) \\
 &= g(x) (\partial_i (g(x)^{-1} s) + A_i g(x)^{-1} s)(x) \\
 &= [g(\partial_i g^{-1}) s + g g^{-1} \partial_i s + g A_i g^{-1} s](x) \\
 &= [\partial_i + g A_i g^{-1} + g \partial_i g^{-1}] s(x)
 \end{aligned}$$

So

$$D'_i = \partial_i + A'_i$$

where

$$A'_i = g A_i g^{-1} + g \partial_i g^{-1}$$

When  $G$  is abelian (as in electromagnetism) we just get

$$A'_i = A_i + g \partial_i g^{-1}.$$

10 May 2005  
(but really part of 5 May lecture)

5) Curvature. If  $\alpha: P \xrightarrow{\sim} P$  is a gauge transformation then the covariant derivative operator

$$D_v: \Gamma(E) \rightarrow \Gamma(E)$$

will transform into

$$D'_v = \alpha D_v \alpha^{-1}$$

where  $\alpha: \Gamma(E) \rightarrow \Gamma(E)$  stands for the action of  $\alpha$  on sections of  $E$ .

The curvature of our original connection  $A$  is

$$F(v, w) = [D_v, D_w] - D_{[v, w]}$$

so the curvature of the transformed connection  $A'$  is

$$\begin{aligned}
 F'(v, w) &= [D'_v, D'_w] - D'_{[v, w]} \\
 &= [\alpha D_v \alpha^{-1}, \alpha D_w \alpha^{-1}] - \alpha D'_{[v, w]} \alpha^{-1} \\
 &= \alpha [D_v, D_w] \alpha^{-1} - \alpha D_{[v, w]} \alpha^{-1} \\
 &= \alpha F(v, w) \alpha^{-1}
 \end{aligned}$$

If we pick a local trivialization of  $P$ , hence of  $E$ , we can think of  $\alpha$  as a  $G$ -valued function

$$g: U \rightarrow G$$

where  $U \subseteq M$  is the open set over which we trivialized  $P$ .

Then

$$F'(v, w) = g F(v, w) g^{-1}$$

and since we can think of  $F$  as a  $\mathfrak{g}$ -valued 2-form on  $U$ , we're just using the adjoint action of  $G$  on  $\mathfrak{g}$  here. Or, if we pick local coordinates we get

$$F'_{ij} = g F_{ij} g^{-1}$$

or

$$F'_{ij} = \text{Ad}(g) F_{ij}$$

confirming the claim that  $F$  is an  $\text{Ad}(P)$ -valued 2-form, where

$$\text{Ad}(P) = P \times_{\mathfrak{g}} \mathfrak{g}$$

where  $G$  acts on  $\mathfrak{g}$  via the adjoint rep.