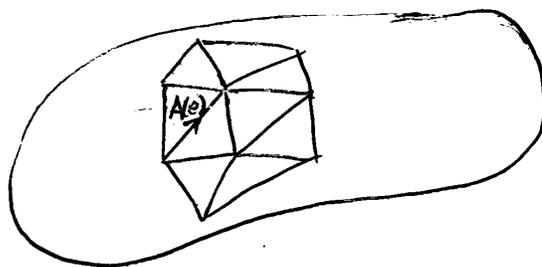


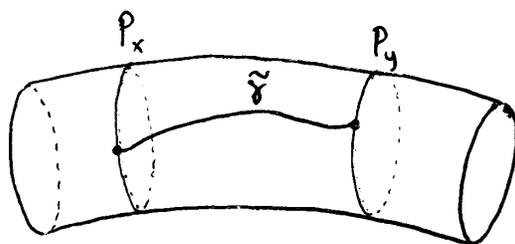
10 May 2005

EF Theory (aka BF theory)

This is a field theory that generalizes the Dijkgraaf-Witten TQFT from a finite group G to a Lie group. The Dijkgraaf-Witten TQFT was all about flat connections on a triangulated manifold:



These assigned to each edge e a group element $A(e)$. This was naive: now we know that a connection on a principal G -bundle over M assigns to any path $\gamma: [0,1] \rightarrow M$ a "parallel transport" map or holonomy:



$H(A, \gamma): P_x \rightarrow P_y$ where $\gamma(0) = x$ $\gamma(1) = y$ and $H(A, \gamma)$ is defined as follows. For any $g \in P_x$ we

take the unique horizontal lift $\tilde{\gamma}$ of γ s.t. $\tilde{\gamma}(0) = q$. Then let

$$H(A, \gamma)q = \tilde{\gamma}(1) \in P_y,$$

i.e. the result of parallel transporting q along γ .

How is

$$H(A, \gamma): P_x \rightarrow P_y$$

like the group element $A(e) \in G$ that was our concept of parallel transport in the discrete context? Since P_x & P_y are G -torsors, we can choose G torsor isomorphisms

$$P_x \cong G$$

$$P_y \cong G$$

and then describe $H(A, \gamma)$ as a map

$$H(A, \gamma): G \rightarrow G$$

& get a group element

$$H(A, \gamma)1 \in G$$

which we can call $A(e)$ when our path γ goes along the edge e . How and why can we reconstruct $H(A, \gamma)$ from this group element $H(A, \gamma)1$?

Claim:

$$H(A, \gamma)(g) = (H(A, \gamma)1)g$$

One proof: $H(A, \gamma): P_x \rightarrow P_y$ is a morphism of (right!) G -torsors,

so

$$H(A, \gamma)(qg) = (H(A, \gamma)q)g \quad q \in P_x$$

Then, if we identify P_x & P_y with G & picking $g=1$ we get

$$H(A, \gamma)(g) = (H(A, \gamma)1)g.$$

So the principal bundle approach to connections does give a group element as the holonomy along a path, but only after choosing isos. $P_x \cong G$ $P_y \cong G$,

(Note: choosing an isomorphism $P_x \cong P_y$ is not enough unless G is abelian.)

Next question: given a G -connection on the triangulated manifold M ,

$$\hat{A}: E \rightarrow G$$

(where E is the set of edges) can we find a connection \hat{A} on some principal G -bundle $P \rightarrow M$ such that

$$\hat{A}(e) = H(\hat{A}, \gamma)$$

where $\gamma: [0, 1] \rightarrow M$ is any path running along e ? Does our choice of P need to depend on \hat{A} ? \hat{A} won't be at all unique. Also: note that $\hat{A}(e) \in G$ but $H(\hat{A}, \gamma): P_x \rightarrow P_y$ is not in G ; we can think of it as being in G only after choosing torsors isomorphisms

$$P_x \cong G \quad P_y \cong G.$$

So let's do this first: for every vertex $x \in V$ of our triangulated manifold, choose an iso. $P_x \cong G$.

Now the question makes sense: given \hat{A} can we find P, \tilde{A} s.t.

$$\hat{A}(e) = H(\tilde{A}, \gamma_e) \quad \forall e \in E$$

where γ_e goes along e ?

Or: given P & \hat{A} , can we find \tilde{A} ? No, this is too hard! Let's consider $G = \mathbb{Z}/2, M = S^1$.



trivial P



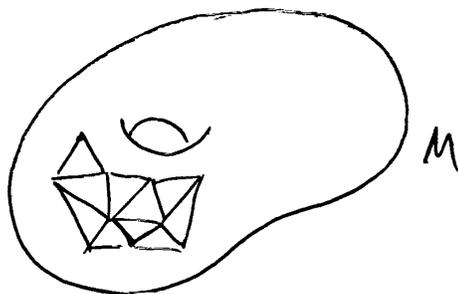
nontrivial P

These bundles each just have one connection, since G is discrete. In the first case we only get \tilde{A} that gives $H(\tilde{A}, e) = 1 \in \mathbb{Z}/2$; in the second case we only get $H(\tilde{A}, e) = -1 \in \mathbb{Z}/2$.

12 May 2005

Discrete vs. Smooth Connections

We have 2 concepts of a connection when G is a Lie group (e.g. a finite group) and M is a smooth manifold equipped with a triangulation Δ .



In the last two quarters, we discussed "discrete" connections; this quarter we're discussing "smooth" connections:

- A discrete connection is a map

$$\hat{A} : E \rightarrow G$$

where E is the set of edges of Δ . Let $\mathcal{A}(\Delta) = G^E$ be the set of discrete connections.

- A smooth connection is an element

$$A \in \Omega^1(P, \mathfrak{g})$$

satisfying some conditions we've discussed. This depends on a choice of principal G -bundle $P \rightarrow M$. Let $\mathcal{A}(P)$ be the set of smooth connections.

The discrete connections are beloved by lattice gauge theorists, TQFT people, & loop quantum graviters. The smooth connections are used in general relativity, perturbative Yang-Mills theory, & differential geometry.

If we trivialize a principal G -bundle $P \rightarrow M$ at the vertices of Δ (i.e., choose a G -torsor isomorphism $P_x \cong G$ for each vertex $x \in V$ of Δ), we saw there's a map

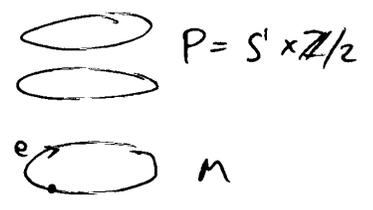
$$\wedge : A \longmapsto \hat{A}$$

sending $A \in \mathcal{A}(P)$ to $\hat{A} \in \mathcal{A}(\Delta)$ such that

$$\hat{A}(e) = H(A, \gamma_e)$$

where $\gamma_e : [0, 1] \rightarrow M$ is any smooth path tracing out the edge e . We've seen:

- 1) \wedge is far from 1-1, since we can change $A \in \mathcal{A}(P)$ to some A' that differs only away from the edges of Δ .
- 2) \wedge is not always onto; e.g. if $G = \mathbb{Z}_2$ & $M = S^1$:



we can only get $1 \in \mathbb{Z}_2$ as the holonomy around e , never -1 .

We can wessel around 2) more easily than 1). We've seen that for $G = \mathbb{Z}_2$, $M = S^1$ that we can get every discrete

connection is \hat{A} where $A \in \mathcal{A}(P)$ for some principal G -bundle P : we use the nontrivial P :



if we want $\hat{A}(e) = -1 \in \mathbb{Z}_2$. If G is a discrete group, P is a covering space, & we can use the theory of covering spaces to show:

Thm: If G is discrete, $\mathcal{A}(P)$ consists of one point, but every discrete connection is of the form \hat{A} for $A \in \mathcal{A}(P)$ for some P .

So in this case, for each P

$$\wedge : \mathcal{A}(P) \longrightarrow \mathcal{A}(\Delta)$$

is 1-1. In fact

$$\bigcup_{\substack{[P] \text{ iso. classes} \\ \text{of principal } G\text{-bundles}}} \mathcal{A}(P) \longrightarrow \mathcal{A}(\Delta)$$

is 1-1 and onto. \blacksquare

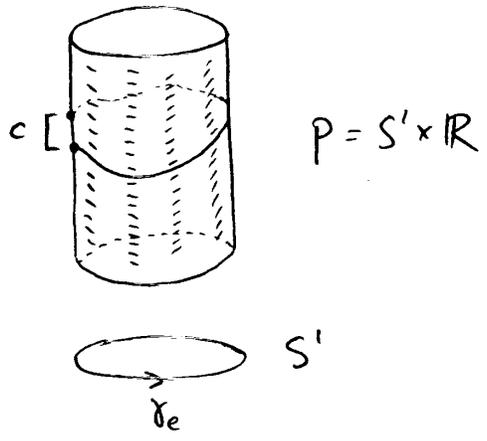
In the Dijkgraaf-Witten model, we computed $Z(M)$ as a sum over discrete connections $\hat{A} \in \mathcal{A}(\Delta)$. The above theorem says this is a sum over isomorphism classes of principal G -bundles. This is an extreme version of ordinary gauge

theory (Yang-Mills theory) in which path integrals involve both an integral over $A(P)$ and a sum over iso classes of P (as seen by 't Hooft in his work on instantons).

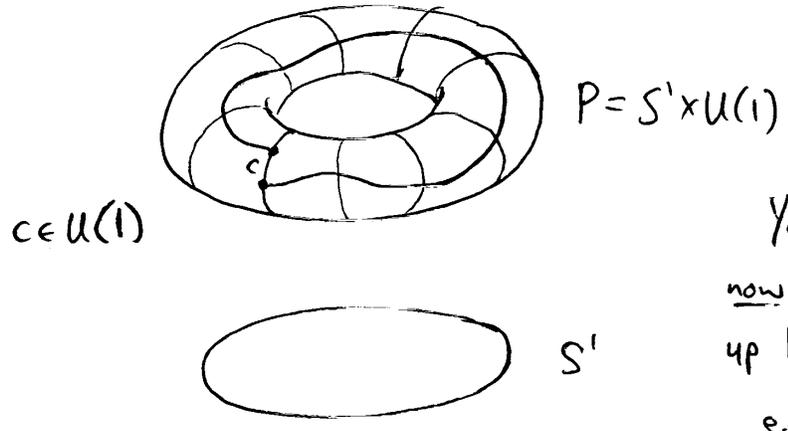
There are other ways to weasel around the fact that

$$\wedge : A(P) \rightarrow A(\Delta)$$

is not onto. Sometimes it is onto, e.g. if $M=S^1$ and $G=\mathbb{R}$:



For any $c \in \mathbb{R}$ we can find a connection A on P such that $H(A, \gamma_e) = c \in \mathbb{R}$. (Note \wedge is far from 1-1 but c determines A up to gauge equivalence.)
How about $G = U(1)$?



Yes, it still works! (But now c does not determine A up to gauge equivalence!

e.g. !)